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On the Error of Hermite Quadrature: An Elementary Proof

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The paper presents an elementary derivation of error estimates for Hermite quadrature rules, which approximate definite integrals using both function and derivative values. Building on ideas from reverse integration by parts, exact errors and explicit error bounds are obtained for the Hermite quadrature. The proposed approach requires only the boundedness of the n-th derivative of the integrand, unlike classical results that demand the boundedness of $2n$-th derivative. Furthermore, numerical examples illustrate that the new bounds often outperform existing estimates, highlighting the practical advantages of the elementary technique for error estimation.

Introduction

Numerical integration is a cornerstone of scientific computing, with the Newton–Cotes formulas being among the most widely used techniques. These methods approximate the integral of a function using only its values at selected points in the interval. The general closed Newton–Cotes formula for approximating the integral of $f(x)$ on $[a,\ b]$ takes the form [**1**]:

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i) + \text{error}, \tag{1}$$

where w_i are the weights, $\{x_i\}_{i=0}^n$ are $(n+1)$ equally spaced points in the interval $[a,\ b]$ with $x_0 = a,\ x_n = b$, and it is assumed that $f(x_i)$ are known at these points. For instance, the case with $n = 1$ is referred to as the “Trapezoidal Rule”, and $n = 2$ corresponds to “Simpsons rule”. A conventional derivation of the error estimates for Trapezoidal and Simpsons rule is based on Newton’s divided differences formulae and the integral mean value theorem [**2**]. A much simpler technique, relying on reverse integration by parts, was developed by Cruz et al. [**3**] to derive error bounds for the

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trapezoid rule. Subsequent work extended this approach to obtain analogous estimates for Simpson's rule [**4, 5**]. This method has gained attention because of it's simplicity, requiring only an elementary background in calculus to establish the error bounds. Moreover, the work resulted in error bounds requiring boundedness of only lower-order derivatives in contrast to earlier bounds that demanded boundedness of higher-order derivatives.

In this work we look at the scenario where in addition to function evaluations $f(x_i)$, one also has access to derivative information at these points. This leads to a natural question:

Can we construct a quadrature rule based on Hermite interpolation, which uses both function and derivative values, to yield a better approximation of the integral?

In particular, we analyse the following quadrature rule:

$$\int_a^b f(x)\,dx = \int_a^b H_n(x)\,dx + \text{error}, \tag{2}$$

where $H_n(x)$ is the two-point Hermite Interpolating Polynomial constructed using the following information on the function and its $n-1$ derivative values: $f(a),\ f'(a), \dots$ $f^{(n-1)}(a),\ f(b),\ f'(b), \dots, f^{(n-1)}(b)$.

To motivate why (2) could provide a better approximation for the integral in comparison to (1), let's consider an example. Consider the function $f(x) = x^2 \sin(x)$ on the interval $[0,\ \pi]$ as shown in Figure 1. Given $f(0)$, $f(\pi)$, $f'(0)$ and $f'(\pi)$ our aim is to approximate the area under the curve (shaded region in Figure 1) in two different ways: i) using Newton-Cotes formulae (1); and ii) using Hermite quadrature (2). We have the following estimates for the integral:

$$\begin{aligned}
&\int_0^\pi f(x)\,dx = \pi^2 - 4, \quad \text{(true estimate)},\\
&\int_0^\pi f(x)\,dx \approx \frac{\pi - 0}{2}\left(f(0) + f(\pi)\right) = 0, \quad \text{(trapezoidal rule)},\\
&\int_0^\pi f(x)\,dx \approx \int_0^\pi H_2(x)\,dx = \int_0^\pi -(x^3 - \pi x^2)\,dx = \frac{\pi^4}{12} \quad \text{(Hermite)},
\end{aligned} \tag{3}$$

where $H_n(x)$ is the two-Point Hermite interpolating polynomial [**6**]. Note that from (3) the error between the trapezoidal rule estimate and the true estimate is $\pi^2 - 4 \approx 5.869$, whereas the error between the Hermite quadrature estimate and the true estimate is approximately 2.247. It is clear from (3) that the Hermite-based quadrature provides a noticeably more accurate approximation. This improvement is due to its use of both function and derivative values, which allows it to capture more information about the function's behavior over the interval. Therefore in scenarios where derivative information is available/easy to compute, Hermite-based quadrature rule present a compelling alternative to classical Newton–Cotes methods.

Note that in the example considered above, the true estimate for the integral was available, which allowed us to compute the error for different quadrature rules. However, if the true estimate of the integral is unavailable, can we derive an estimate (upper bound) for the error in (2)?

An expression for the error in (2) is available in [**6, 2**]. Lampret et al. [**7**] considered a quadrature rule referred to as the "composite Hermite's rule" which applies when both function values $f(a),\ f(a + \frac{b-a}{k}),\ f(a + 2\frac{b-a}{k}), \dots, f(b)$ and the endpoint derivatives $f'(a),\ f'(b)$ are available, where k is the number of sub-intervals in $[a,\ b]$. They also provided an expression for the error in composite Hermite's rule. Note that

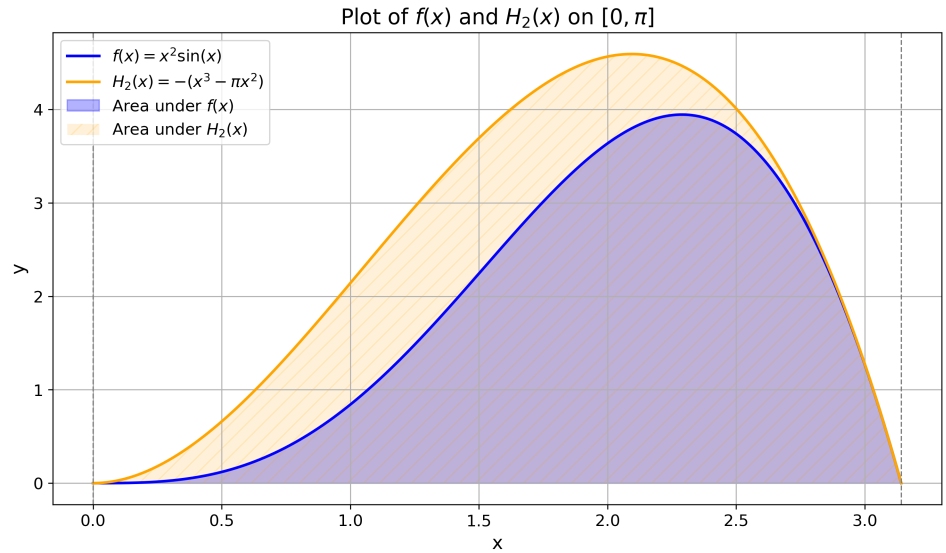

Figure 1 Plot of function $f(x) = x^2 \sin(x)$ with the shaded region denoting the area under the curve between $a = 0$ and $b = \pi$. Note that $\int_a^b f(x)\, dx$ = Area under the curve. The curve corresponding to trapezoidal rule is identically 0 and therefore does not appear in the plot.

the case of $k = 1$ in Lampret et al. [**7**] corresponds to taking $n = 2$ in (2) and the resulting error bounds agree with those in [**6, 2**]. However, it is noteworthy that these error bounds require boundedness of $2n-$th derivative of the integrand.

To the authors best knowledge, the use of elementary techniques such as "reverse integration by parts" for estimating the error in (2) has not been previously explored. While earlier works applied a "reverse integration by parts" formulae once to derive the error for trapezoidal Rule [**3**], in this work we show that applying such a formulae multiple times could indeed not only yield the quadrature rule in (2) but also an explicit expression for it's error. In particular, using the Beta function [**8**] and the General Leibniz Rule [**9**], we derive a novel simplified expression for the integral of the Hermite interpolating polynomial. Comparing this expression with the result of applying reverse integration by parts formulae n times, we obtain an explicit formulae for the error in (2). Moreover, we will demonstrate that the resulting error bounds require only boundedness of the $n-$th derivatives of the integrand, in contrast to the conventional bounds that requires boundedness of $2n-$th derivative.

Hermite Interpolating function and its Integral

To begin, let's consider integrating the Hermite interpolating function $H_n(x)$. Let $f(x)$ be a function defined on the interval $[a, b]$, where $a < b$. If the following information on the function and its $n-1$ derivative values: $f(a),\ f'(a), \ldots f^{(n-1)}(a)$, $f(b),\ f'(b), \ldots f^{(n-1)}(b)$ are available, then $f(x)$ can be approximated by the two-point hermite interpolation polynomial, denoted as $H_n(x)$. The two-point hermite interpolation polynomial is defined as follows [**6, 2**]:

Definition 1 (Two-Point Hermite Interpolating Polynomial)**.** Let a and b be distinct.

The Hermite interpolating polynomial is defined by:

$$H_n(x) = (x-a)^n \sum_{k=0}^{n-1} \frac{B_k (x-b)^k}{k!} + (x-b)^n \sum_{k=0}^{n-1} \frac{A_k (x-a)^k}{k!}, \\ \text{with,} \quad A_k = \frac{d^k}{dx^k}\left[\frac{f(x)}{(x-b)^n}\right]_{x=a}, B_k = \frac{d^k}{dx^k}\left[\frac{f(x)}{(x-a)^n}\right]_{x=b} \tag{4}$$

where $H_n(x)$ satisfies

$$H_n(a) = f(a),\ H_n'(a) = f'(a), ..., H_n^{(n-1)}(a) = f^{(n-1)}(a), \\ H_n(b) = f(b),\ H_n'(b) = f'(b),\ ..., H_n^{(n-1)}(b) = f^{(n-1)}(b).$$

The corresponding error in interpolation is given by [**6, 2**]:

$$f(x) - H_n(x) = \frac{f^{(2n)}(\epsilon)}{(2n)!}\left((x-a)(x-b)\right)^n \tag{5}$$

where $\epsilon \in [a, b]$.

An upper bound for the error in (2) **from literature:** Note that by integrating (5), (2) can now be written as:

$$\int_a^b f(x)\,dx = \int_a^b H_n(x)\,dx + \int_a^b \frac{f^{(2n)}(\epsilon)}{(2n)!}\left((x-a)(x-b)\right)^n\,dx, \tag{6}$$

and a corresponding upper bound for the error in numerical integration can be computed as follows:

$$|\text{error}| = \left|\int_a^b \frac{f^{(2n)}(\epsilon)}{(2n)!}(x-a)^n(x-b)^n\,dx\right| \le \int_a^b \left|\frac{f^{(2n)}(\epsilon)}{(2n)!}\right| |(x-a)^n(x-b)^n|\,dx \\ \le \frac{N}{(2n)!}\int_a^b |(x-a)^n(x-b)^n|\,dx = N(b-a)^{2n+1}\frac{(n!)^2}{(2n)!(2n+1)!} = \epsilon_1, \tag{7}$$

where we assumed that there exists an N **such that** $\left|f^{(2n)}(x)\right| \le N$**,** $\forall x \in [a,\ b]$. However, note that in order to compute (7), it is necessary for $f(x)$ to have its $(2n)^{th}$ derivative bounded on $[a,\ b]$ [**6, 2**]. Consequently, for certain functions such as $x^{3/2}$ on the interval $[-1,\ 1]$, a standard error bound of the form (7) may not be computable. In such cases, one must resort to other techniques to arrive at a computable error estimate.

In our work, we show that by relying on elementary techniques, one can derive a novel error estimate (upper bound for the error) in (2) *that only requires* $\left|f^{(n)}(x)\right| \le M$, $\forall x \in [a,\ b]$ *in comparison to* (7) *which requires* $\left|f^{(2n)}(x)\right| \le N$, $\forall x \in [a,\ b]$.

To that end, we will first attempt to write $\int_a^b H_n(x)\,dx$ in (23) in the following form:

$$\int_a^b H_n(x)\,dx = \sum_{j=0}^{n-1} w_j^a \times f^{(j)}(a) + \sum_{j=0}^{n-1} w_j^b \times f^{(j)}(b), \tag{8}$$

where $f^{(j)}(a)$ denotes the j^{th} derivative of $f(x)$ evaluated at $x = a$, and $w_j^a,\ w_j^b$ are the weights to be determined.

Proposition 1. Consider the Hermite interpolating polynomial in (4) for a given n. Then the weights $w_j^a,\ w_j^b$ in (8) are given by:

$$w_j^a = (b-a)^{j+1} n \sum_{k=j}^{n-1} \binom{k}{j} \frac{(n+k-j-1)!}{(n+k+1)!},$$

$$w_j^b = (-1)^j (b-a)^{j+1} n \sum_{k=j}^{n-1} \binom{k}{j} \frac{(n+k-j-1)!}{(n+k+1)!},$$

where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$.

Proof. We begin by integrating (2) as follows:

$$\begin{aligned}\int_a^b H_n(x)\, dx &= \underbrace{\int_a^b (x-a)^n \sum_{k=0}^{n-1} \frac{B_k (x-b)^k}{k!}\, dx}_{I} \\ &\quad + \underbrace{\int_a^b (x-b)^n \sum_{k=0}^{n-1} \frac{A_k (x-a)^k}{k!}\, dx}_{II}\end{aligned} \tag{9}$$

Let us consider the term (I) in (9). Note that we have:

$$\int_a^b (x-a)^n \sum_{k=0}^{n-1} \frac{B_k (x-b)^k}{k!}\, dx = \sum_{k=0}^{n-1} \frac{B_k}{k!} \int_a^b (x-a)^n (x-b)^k\, dx \tag{10}$$

$$= \sum_{k=0}^{n-1} \frac{B_k}{k!} \int_0^1 t^n (b-a)^n (b-a)^k (t-1)^k (b-a)\, dt, \tag{11}$$

$$= \sum_{k=0}^{n-1} \frac{B_k (b-a)^{n+k+1}}{k!} \int_0^1 t^n (t-1)^k dt \tag{12}$$

$$= \sum_{k=0}^{n-1} \frac{B_k (b-a)^{n+k+1}}{k!} (-1)^k B(n+1, k+1) \tag{13}$$

$$= \sum_{k=0}^{n-1} \frac{B_k (b-a)^{n+k+1}}{k!} (-1)^k \frac{n!k!}{(n+k+1)!}, \tag{14}$$

where we considered the transformation $x = t(b-a)+a$ in going from (10) to (11) and used the definition of Beta function [**8**], $B(n+1,\ k+1)$ in (13) and its value in (14). A similar computation for the term (II) in (9) yields:

$$\int_a^b (x-b)^n \sum_{k=0}^{n-1} \frac{A_k (x-a)^k}{k!} dx = \sum_{k=0}^{n-1} \frac{A_k (b-a)^{n+k+1}}{k!} (-1)^n \frac{n!k!}{(n+k+1)!}. \tag{15}$$

We will now turn our attention to simplifying the terms A_k and B_k from (4). Using the General Leibniz Rule [**9**], we have:

$$\begin{aligned} B_k &= \frac{d^k}{dx^k}\left[\frac{f(x)}{(x-a)^n}\right]_{x=b} \\ &= \left[\sum_{j=0}^{k}\binom{k}{j} f^{(j)}(x)\frac{d^{k-j}}{dx^{k-j}}\left[\frac{1}{(x-a)^n}\right]\right]_{x=b} \\ &= \sum_{j=0}^{k} f^{(j)}(b)\left[\binom{k}{j}\frac{(-1)^{k-j}(n+k-j-1)!}{(n-1)!(b-a)^{n+k-j}}\right]. \end{aligned} \tag{16}$$

Following a similar procedure for A_k in (4) yields:

$$A_k = \sum_{j=0}^{k} f^{(j)}(a)\left[\binom{k}{j}\frac{(-1)^{k-j}(n+k-j-1)!}{(n-1)!(a-b)^{n+k-j}}\right]. \tag{17}$$

Substituting (16), (17) back in (14) and (15) respectively, and recomputing (9) yields:

$$\begin{aligned} \int_a^b H_n(x) &= \sum_{j=0}^{n-1} f^{(j)}(b)\left[(-1)^j(b-a)^{j+1} n\sum_{k=j}^{n-1}\binom{k}{j}\frac{(n+k-j-1)!}{(n+k+1)!}\right] \\ &+ \sum_{j=0}^{n-1} f^{(j)}(a)\left[(b-a)^{j+1} n\sum_{k=j}^{n-1}\binom{k}{j}\frac{(n+k-j-1)!}{(n+k+1)!}\right] \end{aligned} \tag{18}$$

This concludes the proof.

■

An elementary approach to error estimation in (2)

With the results of proposition 1 in hand, we will now discuss the elementary approach used to derive the error estimate in 2. To that end, let's recall the reverse integration by parts formulae below:

$$\int_a^b f(x)\,dx = f(b)(b+c) + f(a)(-a-c) - \int_a^b f'(x)(x+c)\,dx$$

where $c \in \mathbb{R}$. If one repeats the reverse integration by parts process one more time we get:

$$\begin{aligned} \int_a^b f(x)\,dx &= f(b)(b+c) + f(a)(-a-c) + f'(b)\left(-\frac{1}{2}(b+c)^2 - \delta_0\right) \\ &+ f'(a)\left(\frac{1}{2}(a+c)^2 + \delta_0\right) + \int_a^b f''(x)\left(\frac{1}{2}(x+c)^2 + \delta_0\right)\,dx, \end{aligned} \tag{19}$$

where $\delta_0 \in \mathbb{R}$. More generally, if one repeats the procedure n number of times we have:

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{j=0}^{n-1} f^{(j)}(b)\left[(-1)^j\left(\frac{(b+c)^{(j+1)}}{(j+1)!}+\sum_{i=0}^{j-1}\delta_i\frac{b^{j-1-i}}{(j-1-i)!}\right)\right]\\ &+\sum_{j=0}^{n-1} f^{(j)}(a)\left[(-1)^{j+1}\left(\frac{(a+c)^{(j+1)}}{(j+1)!}+\sum_{i=0}^{j-1}\delta_i\frac{a^{j-1-i}}{(j-1-i)!}\right)\right]\\ &+\int_a^b(-1)^n f^{(n)}(x)\left(\frac{(x+c)^{(n)}}{n!}+\sum_{i=0}^{n-2}\delta_i\frac{x^i}{i!}\right)dx,\end{aligned} \tag{20}$$

where we assumed that $f(x) \in C^n[a,\ b]$ * and, $c,\ \delta_0, \dots \delta_{n-2} \in \mathbb{R}$. Now, the central idea of error estimation is as follows: Consider (20) and (18). If one is able to find constants $c,\ \delta_0, \dots \delta_{n-1}$ in (20) such that the coefficients of $f^{(j)}(a)$ and $f^{(j)}(b)$ match exactly for both (20) and (18), then it is clear that the error in the Hermite interpolation-based quadratue in (2) is given by:

$$\begin{aligned}\text{error} &= \int_a^b(-1)^n f^{(n)}(x)\left(\frac{(x+c)^{(n)}}{n!}+\sum_{i=0}^{n-2}\delta_i\frac{x^i}{i!}\right)dx\\ &\leq M\int_a^b\left|\frac{(x+c)^n}{n!}+\sum_{i=0}^{n-2}\delta_i\frac{x^i}{i!}\right|dx=\epsilon_2.\end{aligned} \tag{21}$$

where we used the assumption $\left|f^{(n)}(x)\right| \leq M,\ \forall x \in [a,b]$. In this work, we will demonstrate this procedure for the particular cases $n=2$ and $n=3$ in definition 4. First we will consider the case $n=2$ and the result is summarized in Theorem 1.

Theorem 1. *Consider the case of $n=2$ in definition 4 where we assume that $f(a),\ f(b),\ f'(a),\ f'(b)$ are available. Assume that $f(x) \in C^2[a,\ b]$, and $|f''(x)| \leq M,\ \forall x \in [a,b]$ then one has:*

$$\left|\int_a^b f(x)dx-\int_a^b H_2(x)dx\right| \leq M \times \frac{(b-a)^3(\sqrt{3})}{54}=\epsilon_2. \tag{22}$$

Proof. For $n=2$, (18) can be simplified as:

$$\begin{aligned}\int_a^b H_2(x)\,dx &= f(b)\left(\frac{b-a}{2}\right)+f(a)\left(\frac{b-a}{2}\right)\\ &+f'(b)\left(-\frac{(b-a)^2}{12}\right)+f'(a)\left(\frac{(b-a)^2}{12}\right)\end{aligned} \tag{23}$$

Matching the coefficients between the reverse integration by parts formulae (19) and the integral of the Hermite interpolation (23) we get the following set of equations involving unknowns $c,\ \delta_0$ as follows:

$$b+c=\frac{b-a}{2}, \quad -a-c=\frac{b-a}{2},$$

*$C^n[a,\ b]$ denotes the set of all functions that are n times continuously differentiable on the closed interval $[a,\ b]$

$$-\frac{1}{2}(b+c)^2 - \delta_0 = -\frac{(b-a)^2}{12}, \quad \frac{1}{2}(a+c)^2 + \delta_0 = \frac{(b-a)^2}{12}.$$

It is easy to see that $c = -\frac{a+b}{2}$ and $\delta_0 = -\frac{(b-a)^2}{24}$ satisfy the system of equations above. Therefore, from (19) the error in Hermite interpolation-based quadrature is computed as:

$$\int_a^b f(x)dx - \int_a^b H_2(x)dx = \int_a^b f''(x)\left(\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24}\right) dx$$

Now using the assumption that $|f''(x)| \leq M$ on $[a, b]$ we have:

$$\left|\int_a^b f(x)dx - \int_a^b H_2(x)dx\right| \leq M \int_a^b \left|\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24}\right| dx. \quad (24)$$

Now the integral in (24) can be computed as*:

$$\int_a^b \left|\frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{24}\right| dx = \frac{(b-a)^3(\sqrt{3})}{54}. \quad (25)$$

Therefore, we have:

$$\left|\int_a^b f(x)dx - \int_a^b H_2(x)dx\right| \leq M \times \frac{(b-a)^3(\sqrt{3})}{54},$$

thereby concluding the proof. ■

Let us now derive an error bound for the case $n = 3$ in definition 4.

Theorem 2. *Consider the case of $n = 3$ in definition 4 where we assume that $f(a),\ f(b),\ f'(a),\ f'(b),\ f''(a),\ f''(b)$ are available. Further, assume that $f(x) \in C^3[a,\ b]$, and $|f'''(x)| \leq M$, $\forall x \in [a, b]$ then one has:*

$$\left|\int_a^b f(x)\, dx - \int_a^b H_3(x)\, dx\right| \leq M\frac{13(b-a)^4}{4800} = \epsilon_2. \quad (26)$$

Proof. We follow similar steps as in the proof of Theorem 1. In particular for $n = 3$, (18) can be simplified as:

$$\begin{aligned}\int_a^b H_3(x)\, dx = {} & f(b)\left(\frac{b-a}{2}\right) + f(a)\left(\frac{b-a}{2}\right) \\ & + f'(b)\left(-\frac{(b-a)^2}{10}\right) + f'(a)\left(\frac{(b-a)^2}{10}\right) \\ & + f''(b)\left(\frac{(b-a)^3}{120}\right) + f''(a)\left(\frac{(b-a)^3}{120}\right).\end{aligned} \quad (27)$$

*The integral can be computed using symbolic computation software such as MATLAB (with the Symbolic Math Toolbox) or Python (with the SymPy library)

The reverse integration by parts formulae (20) for $n = 3$ yields:

$$\begin{aligned}\int_a^b f(x)dx &= f(b)(b+c) + f(a)(-a-c) + f'(b)\left(-\frac{1}{2}(b+c)^2 - \delta_0\right)\\ &+ f'(a)\left(\frac{1}{2}(a+c)^2 + \delta_0\right) + f''(b)\left(\frac{1}{6}(b+c)^3 + \delta_0 b + \delta_1\right)\\ &+ f''(a)\left(-\frac{1}{6}(a+c)^3) - \delta_0 a - \delta_1\right)\\ &- \int_a^b f'''(x)\left(\frac{1}{6}(x+c)^3 + \delta_0 x + \delta_1\right)dx\end{aligned} \tag{28}$$

Matching the coefficients between the reverse integration by parts formulae (28) and the integral of the Hermite interpolation (27) we get the following set of equations involving unknowns $c,\ \delta_0,\ \delta_1$ (first comparing coefficients of $f^{(j)}(b)$) as follows:

$$\begin{aligned} b+c &= \frac{b-a}{2},\\ -\frac{1}{2}(b+c)^2 - \delta_0 &= -\frac{(b-a)^2}{10},\\ \frac{1}{6}(b+c)^3 + \delta_0 b + \delta_1 &= \frac{(b-a)^3}{120},\end{aligned} \tag{29}$$

and by comparing the coefficients of $f^{(j)}(a)$ we get:

$$\begin{aligned} -a-c &= \frac{b-a}{2},\\ \frac{1}{2}(a+c)^2 + \delta_0 &= \frac{(b-a)^2}{10},\\ -\frac{1}{6}(a+c)^3 - \delta_0 a - \delta_1 &= \frac{(b-a)^3}{120}.\end{aligned} \tag{30}$$

The solution $c = -\frac{a+b}{2}$, $\delta_0 = -\frac{(b-a)^2}{40}$, and $\delta_1 = \frac{(b-a)^2(a+b)}{80}$ satisfies both the systems (29) and (30). Therefore, from (28) the error in Hermite interpolation-based quadrature is computed as:

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \int_a^b H_3(x)\,dx\right| = \left|\int_a^b f'''(x)\left(\frac{1}{6}(x+c)^3 + \delta_0 x + \delta_1\right)dx\right|\\ &\le M\int_a^b \left|\frac{1}{6}\left(x - \frac{a+b}{2}\right)^3 - \frac{(b-a)^2}{40}x + \frac{(b-a)^2(a+b)}{80}\right|\,dx\\ &\le M \times \frac{13(b-a)^4}{4800},\end{aligned} \tag{31}$$

where we used the assumption that $|f'''(x)| \le M$ on $[a,b]$ and substituted the solution for $c,\ \delta_0,\ \delta_1$. This concludes the proof. ■

Deriving the error bounds for any given n in (2) Note that Theorem 1 and Theorem 2 provides the error bound for $n = 2$ and $n = 3$ respectively. In this section we outline the general procedure for determing the error in (2) for an arbitrary n, i.e for the case when $f(a),\ f''(a), \dots f^{(n-1)}(a),\ f(b),\ f''(b), \dots,\ f^{(n-1)}(b)$ are available. We are also given $\left|f^{(n)}(x)\right| \leq M, \forall x \in [a,b]$. The procedure for error estimation is as follows:

1. Equate the coefficients of $f^{(j)}(a)$ from (20) and (18) to form n equations and estimate the unknowns $c^*,\ \delta_0^*, \dots \delta_{n-2}^*$.
2. Equate the coefficients of $f^{(j)}(b)$ from (20) and (18) to form n equations and again find the unknowns $c^*,\ \delta_0^*, \dots \delta_{n-2}^*$.
3. If the solution $c^*,\ \delta_0^*, \dots \delta_{n-2}^*$ from item 1 and item 2 above are consistent, i.e they give the same solutionThis ended up being very messy to show for a general case. So we left it out for now., then the error estimate is computed from (20) as:

$$\left|\int_a^b f(x)\,dx - \int_a^b H_n(x)\,dx\right| \leq M \int_a^b \left|\frac{(x+c^*)^n}{n!} + \sum_{i=0}^{n-2} \delta_i^* \frac{x^i}{i!}\right| dx, \tag{32}$$

where we used the assumption $\left|f^{(n)}(x)\right| \leq M, \forall x \in [a,b]$.

Remark 1. Note that the new error bound in (32) requires that $\left|f^{(n)}(x)\right| \leq M, \forall x \in [a,b]$ whereas a conventional error bound in (7) requires $\left|f^{(2n)}(x)\right| \leq N, \forall x \in [a,\ b]$.

Numerical demonstration

In this section we compare the error bound (7) from literature with the error bound (21) that we derived in this work by considering different functions $f(x)$ in Table 1 and 2. The "True error" in Table 1 and 2 is defined as:

$$\text{True error} = \left|\int_a^b f(x)\,dx - \int_a^b H_n(x)\,dx\right|,$$

and we used a Newton-Cotes formulae with $n = 10$ in (1) to get an accurate estimate of $\int_a^b f(x)\,dx$. In Table 1 we consider the case $n = 2$, i.e $f(a),\ f(b),\ f'(a),\ f'(b)$ are available to construct the Hermite interpolating polynomial. Similarly, in Table 2 we consider the case $n = 3$, i.e $f(a),\ f(b),\ f'(a),\ f'(b),\ f''(a),\ f''(b)$ are available to construct the Hermite interpolating polynomial. The different functions $f(x)$ used for comparing the error bounds along with the the corresponding Hermite interpolation polynomials $H_2(x)$ and $H_3(x)$ are shown in Figure 2.

TABLE 1: Comparison of error bounds (22) and (7) for the case $n = 2$

$f(x)$	$[a, b]$	True error	Error ϵ_2 in (22)	Error ϵ_1 in (7)
$\cos(8x)\exp(-x)$	$[0,\ 1]$	0.56	**2.02**	5.16
$\tan^{-1}(2x-1)\sin(5x)$	$[0,\ 1]$	0.425	**0.756**	1.80
$\log(1+x)\sin(x)$	$[0,\ 1]$	0.00117	0.0641	**0.0055**
$\cos(\pi x^2)$	$[0,\ 1]$	0.374	**1.266**	2.00

From Table 1 and 2 it is clear that as n increases one could achieve a better error bound ϵ_2 and ϵ_1 since more information on the derivatives of the function are available to construct the Hermite interpolating function. Further, we also observe that ϵ_2 (proposed error estimate) is lower than ϵ_1 (error bound from literature) for most functions

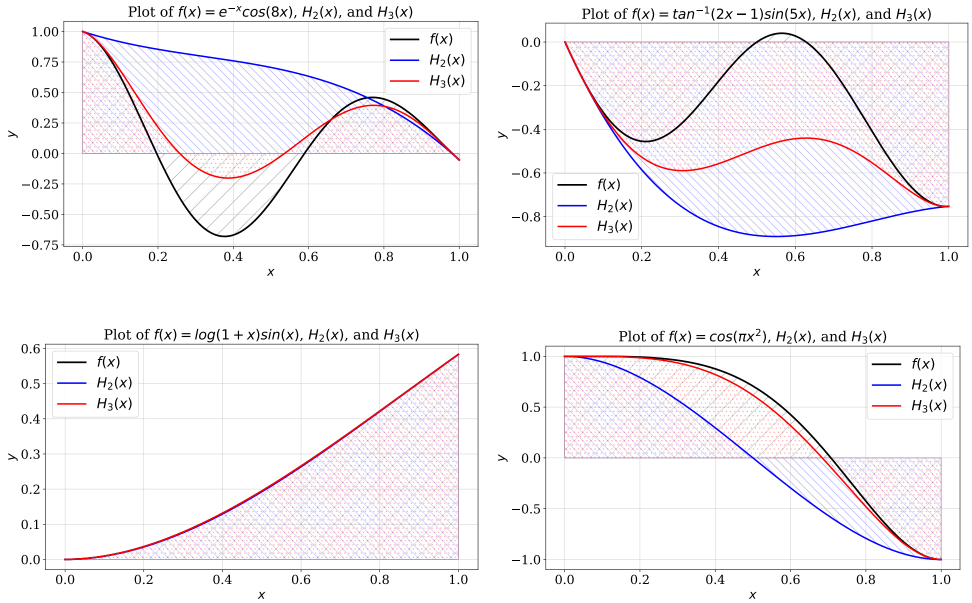

Figure 2 Plot of functions $f(x)$ used in Table 1 and 2 and the corresponding Hermite interpolation polynomials $H_2(x)$ and $H_3(x)$. The shaded region denotes the area under the curve for $f(x)$ and it's approximations $H_2(x)$ and $H_3(x)$.

TABLE 2: Comparison of error bounds (26) and (7) for the case $n = 3$

$f(x)$	$[a, b]$	True error	Error ϵ_2 in (26)	Error ϵ_1 in (7)
$\cos(8x)\exp(-x)$	$[0,\ 1]$	0.15	**1.23**	2.04
$\tan^{-1}(2x-1)\sin(5x)$	$[0,\ 1]$	0.213	**0.44**	0.969
$\log(1+x)\sin(x)$	$[0,\ 1]$	1.27×10^{-4}	0.0081	**0.0010**
$\cos(\pi x^2)$	$[0,\ 1]$	0.045	**0.515**	0.741

we considered. Note that the constant M in (21) and the constant N in (7) plays a key role in determining the efficiency of the error estimators. For the case of $n = 2$, our error estimate (21) will be lower than (7) (the one from literature) if:

$$M \times \frac{(b-a)^3(\sqrt{3})}{54} < N(b-a)^{2\times 2+1}\frac{(2!)^2}{(2\times 2)!(2\times 2+1)!}.$$

This is achieved when:

$$N > \frac{40\sqrt{3}}{3(b-a)^2}M.$$

Similarly for the case of $n = 3$, our error estimate (26) will be lower than (7) (the one from the literature) if:

$$M\frac{13(b-a)^4}{4800} < N(b-a)^{2\times 3+1}\frac{(3!)^2}{(2\times 3)!(2\times 3+1)!}.$$

This is achieved when:

$$N > \frac{273}{3(b-a)^3}M.$$

Thus, knowing the costants N and M allows one to choose between the classical error bound (7) and the bound derived in this work.

Conclusion

In this article, we introduced a novel approach for deriving error estimates for Hermite interpolation-based quadrature rule in (2) using an elementary technique called "reverse integration by parts." The Hermite interpolation-based quadrature rule in (2) is designed to provide a more accurate approximation of the integral $\int_a^b f(x)\,dx$ when both the function and its derivative values are available. In Theorem 1 and Theorem 2 we derived novel error bounds for the cases $n = 2$ and $n = 3$ in (2). We showed that the proposed approach requires only the boundedness of the n-th derivative of the integrand, unlike classical results that demand the boundedness of 2n-th derivative. We also provided numerical evidence in Table 1 and 2 suggesting that the new error bound proposed in this work may offer improvements over existing bounds in the literature.

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