A Stabilized $h$-Adaptive Continuation Method for Double-Diffusive Convection in Porous Media

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• “Double-diffusive” effects occur whenever there are opposing gradients of two diffusing components, each of which affects the local density of a fluid.

• In thermo-solutal convection, the competing components are solute concentration and heat (e.g. a differentially heated layer of sand saturated with brine)

• In this talk, we consider an extremely simple double-diffusive system, and develop a stabilized adaptive finite element method for computing accurate solutions under different parametric regimes.
The equations model slow flow through saturated porous media with buoyancy effects (see e.g. Nield & Bejan, 1992)

\[ \nabla \cdot \mathbf{u} = 0 \\
\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \rho g - \frac{\mu}{K} \mathbf{u} \\
\sigma \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa_T \Delta T \\
\epsilon \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = \kappa_S \Delta S \\
\rho = \rho_0 [1 - \alpha(T - T_0) + \beta(S - S_0)] \]
• \( \mathbf{u} \cdot \hat{n} = 0 \in \Gamma = \Gamma_{\text{top}} \cup \Gamma_{\text{side}} \cup \Gamma_{\text{bot}} \)
• Solute and temperature values fixed on \( \Gamma_{\text{top}} \) and \( \Gamma_{\text{bot}} \)
• No solute or temperature flux from \( \Gamma_{\text{side}} \)
Non-Dim Eqns.

- After manipulation, the standard non-dimensional equations are

\[
\nabla \cdot \mathbf{b} - \Delta p = 0
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{b} - \nabla p) \cdot \nabla T - \Delta T = 0
\]

\[
\frac{\epsilon}{\sigma} \frac{\partial S}{\partial t} + (\mathbf{b} - \nabla p) \cdot \nabla S - \kappa \Delta S = 0
\]

where

\[
\mathbf{b} := (\kappa R_S S - R_T T) \hat{e}_g
\]

(1)

\[
\kappa = 1/Le := \kappa_S/\kappa_T \ll 1
\]

(2)
• The thermal and solutal Rayleigh numbers are defined as

\[ R_T = \frac{gKd\alpha}{\nu\kappa_T} \delta T \]

\[ R_S = \frac{gKd\beta}{\nu\kappa_S} \delta S \]

• Total solute flux is selected as the “Quantity of Interest”.

Rayleigh Numbers
• A steady, quiescent solution with $\hat{e}_g = -\hat{k}$, $0 \leq z \leq 1$ is given by

\[
\begin{align*}
    p_0 &= R_T \left[ T_{\text{bot}} z + \frac{\hat{z}^2}{2} (T_{\text{top}} - T_{\text{bot}}) \right] - \\
    &\quad \kappa R_S \left[ S_{\text{bot}} z + \frac{\hat{z}^2}{2} (S_{\text{top}} - S_{\text{bot}}) \right] \\
    T_0 &= T_{\text{bot}} + z (T_{\text{top}} - T_{\text{bot}}) \\
    S_0 &= S_{\text{bot}} + z (S_{\text{top}} - S_{\text{bot}})
\end{align*}
\]
Numerical Method

- Implicit time discretization (Crank-Nicolson), bilinear Lagrange elements, coupled solve for $p$, $T$, and $S$.

- A simple adaptive continuation algorithm in $\kappa$:
  1. Start with moderate $\kappa$, evolve solution by timestepping from quiescent initial state to steady convecting state.
  2. $h$-adapt mesh to new state.
  3. Decrease $\kappa$, use adapted mesh as new initial condition.
  4. Use steady state equations to converge solution at new $\kappa$.
  5. Return to step 2 until finished.

- Essentially allows accurate parameter space mappings but relies on the solution being stable on the initial coarse grid.
• For small values of $\kappa$, the standard Galerkin method is unstable on coarse grids.

• Solutal Peclet number based on domain height $d = 1$:

$$\max_{\Omega} Pe_S = \max_{\Omega} \frac{|b - \nabla p|}{\kappa}$$

is $O(10^2) - O(10^3)$ for the cases of interest here.

• An SUPG-type stabilized formulation (with nonlinear, scalar $\tau$) is applied to the solute equation only, the $p$ and $T$ equations are solved with standard Galerkin.
• Linear stability diagram for the destabilizing thermal/stabilizing solute configuration

• The diffusivity ratio (in the figure $\tau \equiv \kappa$) affects the mechanism for onset of convection
Numerical Experiment

- Parameter values in the steady onset regime were chosen

\[ R_T = 200 \]
\[ R_S = 160 \]
\[ \epsilon/\sigma = 1/3 \]

- The equations were solved using the continuation algorithm for

\[ 0.01 \leq \kappa \leq 0.1 \]

using both uniform and \( h \)-adaptive grids.
Numerical Experiment (cont.)

- Solute density contours for $\kappa = 0.1$. 

Uniform

Adaptive
Numerical Experiment (cont.)

19,683 dofs

3825 dofs
Numerical Experiment (cont.)

Uniform

Adaptive

- Solute density contours for $\kappa = 0.03$. 
Numerical Experiment (cont.)

19,683 dofs

32,787 dofs
• Solute density $s(y)$ along the centerline $x = 0.5$ for $\kappa = 0.03$. 
Solute density $s(y)$ along the centerline $x = 0.5$ for $\kappa = 0.03$. 
• Solute density $s(y)$ along the quarter-line $x = 0.25$ for $\kappa = 0.03$. 
Convergence

$s(y)$ near $y = 0$

$s(y)$ near $y = 1$

- Solute density $s(y)$ along the quarter-line $x = 0.25$ for $\kappa = 0.03$. 
Solute contours

- Solute density contours and adapted grid for $\kappa = 0.1$. 

3825 dofs
Solute contours

- Solute density contours and adapted grid for $\kappa = 0.095$.

6444 dofs
Solute density contours and adapted grid for $\kappa = 0.09$. 

8154 dofs
Solute contours

- Solute density contours and adapted grid for $\kappa = 0.08$. 

10,452 dofs
Solute contours and adapted grid for $\kappa = 0.07$.  

- Solute density contours and adapted grid for $\kappa = 0.07$. 

15,453 dofs
Solute density contours and adapted grid for \( \kappa = 0.06 \).

- Solute density contours and adapted grid for \( \kappa = 0.06 \).
Solute densities and adapted grid for $\kappa = 0.05$.
Solute contours

- Solute density contours and adapted grid for $\kappa = 0.04$. 

27,342 dofs
Solute density contours and adapted grid for $\kappa = 0.03$.

32,787 dofs
Solute contours

- Solute density contours and adapted grid for $\kappa = 0.025$. 

39,612 dofs
Solute contours

- Solute density contours and adapted grid for $\kappa = 0.02$.

48,513 dofs
Solute density contours and adapted grid for \( \kappa = 0.01 \).

- Solute density contours and adapted grid for \( \kappa = 0.01 \).
• Computed values of $N_S = -\int_{\Gamma_{\text{bot}}} (\nabla S \cdot \hat{n})$ for varying $\kappa$ on different grids
• Comparison of dofs used for different $\kappa$. 
Conclusions and Future Work

• Adaptivity is an attractive technique for handling layers in small $\kappa$ (high $Le$) cases, since the locally varying mesh scales help prevent cell Peclet violations.

• Some form of stabilization must be added to the standard Galerkin method in small $\kappa$ cases. Its use is essential on coarse initial grids.

• This study served as a proof of concept: three-dimensional calculations and additional parametric studies ($\kappa, R_S, R_T, \ldots$) are the final goal.