Problem 1: (10p) Let $H$ be a Hilbert space, and let $A \in \mathcal{B}(H)$.

(a) (5p) Define the spectrum $\sigma(A)$.

(b) (5p) Suppose that $A$ is skew-adjoint and that $\|A\| = 2$. Are there any complex numbers $\lambda$ for which you can say for sure that $A - \lambda I$ is one-to-one and onto?

**Solution:**

(b) Since $A$ is skew-adjoint, you know that if $\text{Re}(\lambda) \neq 0$, then $\lambda \notin \sigma(A)$.

Since $\|A\| = 2$, you know that if $|\lambda| > 2$, then $\lambda \in \rho(A)$.

Consequently, $A - \lambda I$ is necessarily one-to-one and onto if either $|\lambda| > 2$ or if $\text{Re}(\lambda) \neq 0$.

Problem 2: (10p) Let $T \in \mathcal{S}^*(\mathbb{R})$ be defined via $T(\varphi) = \int_{-\infty}^{\infty} \log |x| \varphi(x) \, dx$. Specify the derivative of $T$.

No motivation required.

**Solution:**

The derivative of $T$ is the principal value of $1/x$.

To prove this, note that

$$[DT](\varphi) = -T(\varphi') = \lim_{\varepsilon \searrow 0} \left\{ -\int_{-\infty}^{-\varepsilon} \log |x| \varphi'(x) \, dx - \int_{\varepsilon}^{\infty} \log |x| \varphi'(x) \, dx \right\}$$

$$= \lim_{\varepsilon \searrow 0} \left\{ -[\log |x| \varphi(x)]_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{1}{x} \varphi(x) \, dx - [\log |x| \varphi(x)]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{1}{x} \varphi'(x) \, dx \right\}$$

$$= \lim_{\varepsilon \searrow 0} \left\{ -\log(\varepsilon)\varphi(-\varepsilon) + \int_{-\infty}^{-\varepsilon} \frac{1}{x} \varphi(x) \, dx + \log(\varepsilon)\varphi(\varepsilon) + \int_{\varepsilon}^{\infty} \frac{1}{x} \varphi'(x) \, dx \right\} = PV(1/x)(\varphi),$$

since $\lim_{\varepsilon \searrow 0} \log(\varepsilon)(\varphi(\varepsilon) - \varphi(-\varepsilon)) = 0$. 


Problem 3: (10p) No motivations required for these two problems.

(a) (5p) Let $H$ be a Hilbert space, and let $A \in \mathcal{B}(H)$ be an operator that satisfies $A^2 = A = A^*$. The operator $A$ is neither the zero or the identity operator. Specify $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$.

(b) (5p) Let $H = L^2([0, \infty))$, and let $A \in \mathcal{B}(H)$ be defined by $[Au](x) = \arctan(x) u(x)$. Specify $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$.

Solution:

(a) $A$ is a non-trivial orthogonal projection. As shown in the homework, this means that $\sigma_p(A) = \{0, 1\}$, $\sigma_c(A) = \emptyset$, $\sigma_r(A) = \emptyset$.

(b) $A$ is a multiplication operator so $\sigma(A)$ equals the closure of the range of the function being multiplied. In this case the spectrum is purely a continuum spectrum since there are no stationary points in the range. So $\sigma_p(A) = \emptyset$, $\sigma_c(A) = [0, \pi/2]$, $\sigma_r(A) = \emptyset$.

Problem 4: (10p) Consider the four sequences in $S^*(\mathbb{R})$ given below. Specify which sequences are convergent. If the sequence is convergent, then specify the limit. No motivations required.

(a) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \sin(nx)$.

(b) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \begin{cases} n & \text{when } -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 0 & \text{when } |x| > \frac{1}{n}. \end{cases}$

(c) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \begin{cases} n^2 & \text{when } -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 0 & \text{when } |x| > \frac{1}{n}. \end{cases}$

(d) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!}$.

Solution:

(a) $T_n \rightarrow 0$. We proved this in class.

(a) $T_n \rightarrow 2\delta$. We proved something very similar in class.

(c) Divergent. You can easily prove that $\lim_{n \to \infty} T_n(\varphi) = \lim_{n \to \infty} 2n\varphi(0)$.

(d) Divergent. We have $\lim_{n \to \infty} T_n(x) = e^x$, and $e^x$ is not a tempered distribution. (If you’d like to prove things rigorously, consider $\varphi(x) = \exp(-(1 + x^2)^{1/4})$. Then $\varphi \in S$ and $T_n(\varphi) \rightarrow \infty$.)
Problem 5: (20p) Let $H$ denote the Hilbert space $H = l^2(\mathbb{Z})$. In other words, a doubly indexed vector $x = \{x(n)\}_{n=-\infty}^{\infty}$ belongs to $H$ iff $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$. Define $A \in B(H)$ via:

$$[Ax](n) = x(n + 1) - x(n - 1), \quad n \in \mathbb{Z}.$$ 

Let $F : L^2(\mathbb{T}) \to H$ denote the standard Fourier transform, and let $F^{-1}$ denote its inverse. Define $B = F^{-1}AF$ as an operator on $L^2(\mathbb{T})$.

(a) (5p) Determine the action of $B$ on a function $u = u(t)$ in $L^2(\mathbb{T})$.

(b) (15p) Determine $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$.

Solution:

(a) Consider a function $u = \sum_{n=-\infty}^{\infty} a_n e_n$, where $e_n(x) = e^{inx}/\sqrt{2\pi}$ as usual. Then $Fu = \{a_n\}$ and $AFu = \{a_{n+1} - a_{n-1}\}$. Then

$$[F^{-1}AFu](x) = \sum_{n=-\infty}^{\infty} (a_{n+1} - a_{n-1}) \frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} e^{-ix} a_{n+1} \frac{e^{i(n+1)x}}{\sqrt{2\pi}} - \sum_{n=-\infty}^{\infty} e^{ix} a_{n-1} \frac{e^{i(n-1)x}}{\sqrt{2\pi}} = (e^{-ix} - e^{ix}) u(x) = -2i \sin(x) u(x).$$

(b) Since $A$ and $B$ are unitarily equivalent, their spectra are identical. First note that

$$\langle Bu, v \rangle = \int_{-\pi}^{\pi} -2i \sin(x) u(x) v(x) \, dx = \int_{-\pi}^{\pi} u(x) 2i \sin(x) v(x) \, dx = \langle u, -Bv \rangle,$$

so $B$ is skew-adjoint. This proves that $\sigma_r(B) = \emptyset$ and that $\sigma(B)$ is a subset of the imaginary line.

Let us first search for eigenvalues. Suppose $Bu = \lambda u$. Then

$$(-2i \sin(x) - \lambda) u(x) = 0, \quad \text{a.e.}$$

Since $-2i \sin(x) - \lambda = 0$ except possibly for a set of measure zero, we find that $\sigma_p(B) = \emptyset$.

Set $\Omega = \{ib : b \in [-2, 2]\}$. In other words, $\Omega$ is the range of the function $f(x) = -2i \sin(x)$, and our guess at this point is that $\Omega$ is the continuum spectrum.

Suppose that $\lambda \notin \Omega$. Set $d = \inf \{ |\lambda - z| : z \in \Omega \} = \text{dist}(\lambda, \Omega)$. Since $\Omega$ is closed we know that $d > 0$. Then

$$\| (B - \lambda I)^{-1} u \|^2 = \int_{-\pi}^{\pi} \left| \frac{1}{f(x) - \lambda} u(x) \right|^2 \, dx \leq \int_{-\pi}^{\pi} \frac{1}{d^2} |u(x)|^2 \, dx = \frac{1}{d^2} \|u\|^2$$

so $\| (B - \lambda I)^{-1} \| \leq 1/d < \infty$, which shows that $\lambda \in \rho(B)$.

Suppose that $\lambda = ib \in \Omega$ for some $b \in [-\pi, \pi]$. Let $a \in [-\pi, \pi]$ be such that $f(a) = ib$. Then pick non-negative functions $\varphi_n$ such that $\| \varphi_n \| = 1$, and $\varphi_n(x) = 0$ when $|x - a| \geq 1/n$. Then

$$\| (B - \lambda I) \varphi_n \|^2 = \int_{-\pi}^{\pi} |(f(x) - ib) \varphi_n(x)|^2 \, dx = \int_{a-1/n}^{a+1/n} |f(x) - ib|^2 |\varphi_n(x)|^2 \, dx \leq \frac{8}{3n^3} \|\varphi_n\|^2 = \frac{8}{3n^3},$$

where we used that $|f(x) - ib| = |\int_a^x f'(t) \, dt| \leq 2|x - a|$ since $|f'| \leq 2$. The inequality proven shows that $B - \lambda I$ is not coercive, and consequently cannot have closed range.

$$\sigma_p(A) = \emptyset, \quad \sigma_c(A) = \{ib : b \in [-2, 2]\}, \quad \sigma_r(A) = \emptyset.$$
Problem 6: \((4 \times 5p)\) For each of the four operators defined below, determine whether it is well-defined, and whether it is continuous.

(a) \(A : \mathcal{S}(\mathbb{R}) \to \mathbb{C}\) defined via \(A(\varphi) = \int_{\mathbb{R}} x^2 \varphi(x) \, dx\).

(b) \(B : \mathcal{S}(\mathbb{R}) \to \mathbb{C}\) defined via \(B(\varphi) = \int_{\mathbb{R}} x(\varphi(x))^2 \, dx\).

(c) \(C : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})\) defined via \(\langle C(\varphi) \rangle(x) = x \varphi(x)\).

(d) \(D : \mathcal{S}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})\) defined via \(DT = \partial T\). (Just plain differentiation.)

Solution:

(a) Pick \(\varphi \in \mathcal{S}\). Then \(|A(\varphi)| \leq \int_{\mathbb{R}} \frac{x^2}{(1 + x^2)^2} \, d\varphi(x) \leq \int_{\mathbb{R}} \frac{x^2}{(1 + x^2)^2} \, d\varphi \leq \int_{\mathbb{R}} \frac{x^2}{(1 + x^2)^2} \, dx \varphi \|0,4 = C \parallel \varphi \|0,4\). This proves that \(A\) is well-defined. Next we prove continuity. Suppose that \(\varphi_n \to \varphi\) in \(\mathcal{S}\). Then \(|A(\varphi) - A(\varphi_n)| \leq C \parallel \varphi - \varphi_n \|0,4 \to 0\).

(b) Pick \(\varphi \in \mathcal{S}\). Then \(|B(\varphi)| \leq \int_{\mathbb{R}} \frac{|x|}{(1 + x^2)^2} \, d\varphi(x) \leq \int_{\mathbb{R}} \frac{|x|}{(1 + x^2)^2} \, dx \varphi \|0,2 = C \parallel \varphi \|0,2\). This proves that \(B\) is well-defined. Next we prove continuity. Suppose that \(\varphi_n \to \varphi\) in \(\mathcal{S}\). Set \(M = sup_{\mathcal{S}} \|\varphi_n\|0,0\). Since \((\varphi_n)\) is convergent to \(\varphi\) wrt the uniform norm, we know that \(M < \infty\) and that \(\parallel \varphi \|0,0 \leq M\). Then
\[
|B(\varphi) - B(\varphi_n)| \leq \int_{-\infty}^{\infty} |x| (\varphi(x))^2 - (\varphi_n(x))^2 \, dx = \int_{-\infty}^{\infty} |x| ((\varphi(x) + \varphi_n(x))(\varphi(x) - \varphi_n(x))) \, dx \\
\leq \int_{-\infty}^{\infty} 2M |\varphi(x) - \varphi_n(x)| \, dx = 2M \int_{-\infty}^{\infty} \frac{|x|}{(1 + x^2)^2} (1 + x^2)\varphi(x) - \varphi_n(x) \, dx \\
\leq 2M \int_{-\infty}^{\infty} \frac{|x|}{(1 + x^2)^2} \, dx \varphi - \varphi_n \|0,4 \to 0.
\]

(c) Fix \(\varphi \in \mathcal{S}\). Fix \(\alpha, k \in \mathbb{Z}_+\). Then
\[
\parallel C(\varphi) \|_{\alpha,k} = sup_{x} (1 + x^2)^k/2 |\partial^{\alpha}(\varphi(x))| = sup_{x} (1 + x^2)^k/2 |x\partial^{\alpha} \varphi + \alpha \partial^{\alpha-1} \varphi| \leq M \parallel \varphi \|_{\alpha,k+1} + \alpha \parallel \varphi \|_{\alpha-1,k},
\]
where \(M\) is the finite number given by \(M = sup \frac{|x|}{(1 + x^2)^{(k+1)/2}}\). This inequality proves that \(C(\varphi) \in \mathcal{S}\). Next consider continuity. Suppose that \(\varphi_n \to \varphi\) in \(\mathcal{S}\). Then for any \(\alpha, k \in \mathbb{Z}_+\) we have
\[
\parallel C(\varphi) - C(\varphi_n) \parallel_{\alpha,k} \leq \cdots \leq M \parallel \varphi - \varphi_n \|_{\alpha,k+1} + \alpha \parallel \varphi - \varphi_n \|_{\alpha-1,k} \to 0.
\]

(d) Fix \(T \in \mathcal{S}^*\). We will first prove that \(D(T)\) is a distribution. Fix \(\varphi \in \mathcal{S}\). Then by definition
\[
\langle D(T), \varphi \rangle = -\langle T, \varphi' \rangle.
\]
We proved in class that \(\varphi' \in \mathcal{S}\) so \(D(T)\) evaluates to a finite complex number. To establish that \(D(T)\) is in \(\mathcal{S}^*\), we also need to prove that \(D(T)\) is continuous. This follows from the fact that \(\varphi_n \to \varphi\) in \(\mathcal{S}\) implies that \(\varphi_n' \to \varphi'\) in \(\mathcal{S}\) (also proven in class). So \(D(T)\) is well-defined.

Is the map \(D : \mathcal{S}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})\) continuous? We need to prove that if \(T_n \to T\) in \(\mathcal{S}^*\), then \(D(T_n) \to D(T)\) in \(\mathcal{S}^*\). Suppose that \(T_n \to T\) in \(\mathcal{S}^*\). Fix \(\varphi \in \mathcal{S}\). Then
\[
\langle D(T_n), \varphi \rangle = -\langle T_n, \varphi' \rangle \rightarrow \{\text{Since } T_n \to T \text{ and } \varphi' \in \mathcal{S}\} \rightarrow -\langle T, \varphi' \rangle = \langle D(T), \varphi \rangle.
\]

In summary: All maps are well-defined and continuous.