Homework set 12 — APPM5450, Spring 2017 — Hints

Problem 12.2:

(a) Use that \( A \setminus B = A \cap B^c = (A^c \cup B)^c \).

(b) Split \( B \) into two well-chosen disjoint sets and use additivity.

(c) Split \( A \cup B \) into three well-chosen disjoint sets and use additivity. (I think we did this one in class.)

Problem 12.3: The trick is to write \( \bigcup_{n=1}^{\infty} A_n \) as a disjoint union. For \( n = 1, 2, 3, \ldots \) set \( B_n = A_{n+1} \setminus A_n \). Then

\[
\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left( \bigcup_{n=1}^{\infty} B_n \right),
\]

where there union on the right is a disjoint one. Now use additivity twice:

\[
\mu\left( \bigcup_{n=1}^{\infty} A_n \right) = \mu\left( A_1 \cup \left( \bigcup_{n=1}^{\infty} B_n \right) \right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n)
\]

\[
= \lim_{N \to \infty} \left( \mu(A_1) + \sum_{n=1}^{N} \mu(B_n) \right) = \lim_{N \to \infty} \mu\left( A_1 \cup \left( \bigcup_{n=1}^{N} B_n \right) \right) = \lim_{N \to \infty} \mu(A_N).
\]

For the second part, set \( C = \cap_{n=1}^{\infty} A_n \) and \( C_n = A_n \setminus A_{n+1} \). Then

\[
\mu(A_N) = \mu\left( C \cup \left( \bigcup_{n=N}^{\infty} C_n \right) \right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).
\]

Since \( \infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n) \), we find that

\[
\lim_{N \to \infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,
\]

which completes the proof. For the counterexample, consider \( X = \mathbb{R}^2 \), and \( A_n = \{ x = (x_1, x_2) : |x_2| < 1/n \} \). Then \( \mu(A_n) = \infty \) for all \( n \), but \( \bigcap_{n=1}^{\infty} A_n \) is the \( x_1 \)-axis, which has measure zero.

Problem 12.5: Straight-forward.

Problem 12.7:

**Reflexivity:** It is obvious that \( f(x) = f(x) \) a.e.

**Symmetry:** If \( f(x) = g(x) \) a.e., then obviously \( g(x) = f(x) \) a.e.
Transitivity: Suppose that $f(x) = g(x)$ a.e. and that $g(x) = h(x)$ a.e. Set

$$A = \{ x : f(x) \neq g(x) \}$$

$$B = \{ x : g(x) \neq h(x) \}$$

$$C = \{ x : f(x) \neq h(x) \}.$$ 

We know that $\mu(A) = \mu(B) = 0$, and we want to prove that $\mu(C) = 0$. It is clearly the case that $C \subseteq A \cup B$, and then it follows directly that $\mu(C) \leq \mu(A) + \mu(B) = 0.$