Problem 11.22: Set \( T = \text{sign}(t) \). We seek to prove that \( \hat{T} = \alpha \text{PV}(1/x) \) for some \( \alpha \).

For \( N = 1, 2, 3, \ldots \), set \( T_N = \chi_{[-N,N]} \) \( T \). Then \( T_N \to T \) in \( S^* \) since for any \( \varphi \in S \), we have

\[
(T_n, \varphi) = \int_{-N}^{N} \text{sign}(x) \varphi(x) \, dx \to \int_{-\infty}^{\infty} \text{sign}(x) \varphi(x) \, dx = \langle T, \varphi \rangle.
\]

Since the Fourier transform is a continuous operator on \( S^* \), we know that \( \hat{T} \) is the limit of the sequence \( \langle T_n \rangle \).

Since \( T_N \in L^1 \), we can compute \( \hat{T}_N \) by directly evaluating the integral. We find that

\[
\hat{T}_N(x) = \beta \frac{1 - \cos(Nx)}{x}
\]

for some constant \( \beta \). If \( \varphi \in S \), then

\[
\langle \frac{1 - \cos(Nx)}{x}, \varphi \rangle = \int_{\mathbb{R}} \frac{1 - \cos(Nx)}{x} \varphi(x) \, dx = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{1 - \cos(Nx)}{x} \varphi(x) \, dx = \langle \text{PV}(1/x), \varphi \rangle - \langle \cos(Nx) \text{PV}(1/x), \varphi \rangle.
\]

It follows that formula (1) can be written \( \hat{T}_N(x) = \beta \text{PV}(1/x) - \beta \cos(Nx) \text{PV}(1/x) \).

It remains to prove that \( \cos(Nx) \text{PV}(1/x) \to 0 \) in \( S' \). We find that

\[
\langle \cos(Nx) \text{PV}(1/x), \varphi \rangle = \langle \text{PV}(1/x), \cos(Nx) \varphi \rangle
\]

\[
= \int_{0}^{\infty} \cos(Nx) \frac{1}{x} \varphi(x) \, dx + \int_{-\infty}^{0} \cos(Nx) \frac{1}{x} \varphi(x) \, dx
\]

\[
= \int_{0}^{\infty} \cos(Nx) \frac{\varphi(x) - \varphi(-x)}{x} \, dx.
\]

Now set \( \psi(x) = \frac{\varphi(x) - \varphi(-x)}{x} \). Then \( \psi \) is a continuously differentiable, quickly decaying function on \([0, \infty)\), so we can perform a partial integration to obtain

\[
\left| \int_{0}^{\infty} \cos(Nx) \frac{\varphi(x) - \varphi(-x)}{x} \, dx \right| = \left| \left[ \frac{\sin(Nx)}{N} \psi(x) \right]_{0}^{\infty} - \int_{0}^{\infty} \sin(Nx) \frac{\psi'}{x} \, dx \right| 
\]

\[
\leq \frac{1}{N} \int_{0}^{\infty} |\psi'(x)| \, dx.
\]

If we can prove that \( \int_{0}^{\infty} |\psi'(x)| \, dx < \infty \), we will be done. First note that for \( x \in [0, 1] \), \( \psi(x) = 2\varphi(0) + O(x^2) \), so for \( x \in [0, 1] \), we have \( |\psi'(x)| \leq C_1 \) for some finite \( C_1 \). For \( x \in [1, \infty) \), we have

\[
|\psi'(x)| = \left| \frac{\varphi'(x) + \varphi'(-x)}{x} - \frac{\varphi(x) - \varphi(-x)}{x^2} \right| \leq 2 \frac{||\varphi||_{L^1}}{x^2} + 2 \frac{||\varphi||_{L^2}}{x^2} = \frac{C_2}{x^2}.
\]

and so

\[
\int_{0}^{\infty} |\psi'(x)| \, dx \leq \int_{0}^{1} C_1 \, dx + \int_{1}^{\infty} \frac{C_2}{x^2} \, dx < \infty.
\]