Problem 9.21: Suppose $A \in \mathcal{B}(H)$ is such that
$$\text{Re}(x, Ax) \leq 2\alpha \|x\|^2.$$ 
Prove that the solution $x = x(t)$ of $x'(t) = Ax(t)$ satisfies
$$\|x(t)\| \leq e^{\alpha t} \|x(0)\|.$$

Note: The book may have a typo — the bound seems off by a factor of two. Consider for instance $Ax = 2\alpha x$, then $x(t) = e^{2\alpha t} x(0)$.

Solution: Set $f(t) = \|x(t)\|^2$. Then
$$f'(t) = \frac{d}{dt} (x, x) = (x', x) + (x, x') = (Ax, x) + (x, Ax) = 2\text{Re}(x, Ax) \leq 4\alpha \|x(t)\|^2 = 4\alpha f(t).$$

By the Grönwall inequality, we find
$$\|x(t)\|^2 = f(t) \leq f(0) \exp(\int_0^t 4\alpha ds) = f(0) e^{4\alpha t} = \|x(0)\|^2 e^{4\alpha t}.$$

Extract the square root to obtain the desired bound.

Problem 9.22: Let $A$ be compact and non-negative. Prove that there exists a unique compact non-negative operator $B$ such that $B^2 = A$.

Solution: Since $A$ is self-adjoint and compact, there is an ON-basis $(\varphi_n)_{n=1}^\infty$ of eigen-vectors of $A$. $A \varphi_n = \lambda_n \varphi_n$. We know $|\lambda_n| \to 0$ since $A$ is compact, and $\lambda_n \geq 0$ since $A$ is non-negative.

Existence: Set $B = \sum_{n=1}^\infty \sqrt{\lambda_n} P_n$ where $P_n x = (\varphi_n, x) \varphi_n$. It is easily shown that $B^2 = A$ and that $B$ is compact and non-negative.

Observe that from the construction of $B$, it follows that if $\psi$ is a vector such that $A \psi = \lambda \psi$, then $B \psi = \sqrt{\lambda} \psi$.

Uniqueness: Suppose that $C$ is a non-negative compact operator such that $C^2 = A$. We need to show that $C = B$, where $B$ is the operator constructed above. Since $C$ is compact and self-adjoint, there is an ON-basis $(\psi_n)_{n=1}^\infty$ such that $C \psi_n = \mu_n \psi_n$. Now observe that
$$A \psi_n = C^2 \psi_n = C (\mu_n \psi_n) = \mu_n^2 \psi_n$$
so $\psi_n$ is an eigenvector of $A$ with eigenvalue $\mu_n^2$. It follows that $B \psi_n = \sqrt{\mu_n^2} \psi_n = \mu_n \psi_n = C \psi_n$. (We know that $\sqrt{\mu_n^2} = \mu_n$ since $C$ must be non-negative, which implies that $\mu_n \geq 0$.)
Problem 1: Consider the Hilbert space $H = \mathbb{C}^n$. Let $A \in \mathcal{B}(H)$, let $(e^{(j)})_{j=1}^n$ be the canonical basis, and let $A$ have the representation

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

in the canonical basis. We define the Hilbert-Schmidt norm of $A$ as

$$\|A\|_{\text{HS}} = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$ 

(a) Let $(\varphi^{(j)})_{j=1}^n$ be any ON-basis for $H$. Show that $\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|A\varphi^{(j)}\|^2$.

(b) Show that $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n} \|A\|$ for any $A \in \mathcal{B}(H)$.

(c) Find $G, H \in \mathcal{B}(H)$ such that $\|G\|_{\text{HS}} = \|G\|$ and $\|H\|_{\text{HS}} = \sqrt{n} \|H\|$.

Solution:

(a) Let $r^{(i)}$ denote the $i$th row of $A$. Then

$$\sum_{j=1}^n \|A\varphi^{(j)}\|^2 = \sum_{j=1}^n \sum_{i=1}^n \|(r^{(i)}, \varphi^{(j)})\|^2 = \{\text{Parseval}\} = \sum_{i=1}^n \|r^{(i)}\|^2 = \|A\|^2_{\text{HS}}.$$ 

(b) For any $x$ a simple application of Cauchy-Schwartz yields

$$\|Ax\|^2 = \sum_{i=1}^n \|(r^{(i)}, x)\|^2 \leq \sum_{i=1}^n \|r^{(i)}\|^2 \|x\|^2 = \|A\|^2_{\text{HS}} \|x\|^2.$$ 

It follows that $\|A\| \leq \|A\|_{\text{HS}}$. Next, let $i$ be such that $\|r^{(i)}\| = \max_j \|r^{(j)}\|$. Then

$$\|A\|^2_{\text{HS}} = \sum_{j=1}^n \|r^{(j)}\|^2 \leq n \|r^{(i)}\|^2 = n \|A^* e_i\|^2 \leq n \|A^*\| = n \|A\|,$$

where $e_i$ denotes the $i$th canonical basis vector.

(c) For instance, let $G$ be the matrix consisting of all ones. Then, the singular value decomposition of $G$ is $G = \sqrt{n} gg^*$, where $g = (1,1,1,\ldots,1)/\sqrt{n}$. Consequently, $\|G\| = \sqrt{n}$. It is a trivial computation to show that $\|G\|_{\text{HS}} = \sqrt{n}$.

Next, let $H$ be the identity matrix. Then obviously $\|H\| = 1$ since $\|Hx\| = \|x\|$ for any vector $x$. But $\|H\|_{\text{HS}} = \sqrt{n}$. 


Problem 2: Let $H$ be a separable Hilbert space, and let $A \in \mathcal{B}(H)$. Suppose that $H$ has an ON-basis $(\varphi^{(j)})_{j=1}^{\infty}$ such that

$$
\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 < \infty.
$$

Prove that if $(\psi^{(j)})_{j=1}^{\infty}$ is any other ON-basis, then

$$
\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \|A\psi^{(j)}\|^2.
$$

Solution: Set

$$
\alpha_{ji} = (A\varphi^{(j)}, \psi^{(i)}) = (\varphi^{(j)}, A^* \psi^{(i)})
$$

and

$$
\beta_{ik} = (A^* \psi^{(i)}, \psi^{(k)}) = (\psi^{(i)}, A \psi^{(k)}).
$$

The proof consists of four applications of Parseval:

$$
\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\alpha_{ji}|^2 = \sum_{i=1}^{\infty} \|A^* \psi^{(i)}\|^2 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\beta_{ik}|^2 = \sum_{k=1}^{\infty} \|A \psi^{(k)}\|^2.
$$

Note that the interchanges of summation order are permissible as all terms are non-negative.
Problem 3: Consider the linear space $L = \mathbb{R}^2$. Define for $x = (x_1, x_2) \in L$ the seminorms

$$p_1(x) = |x_1|, \quad p_2(x) = |x_2|.$$ 

Construct for $x \in L$, $j \in \{1, 2\}$, and $\varepsilon \in (0, \infty)$, the sets

$$B_{x,j,\varepsilon} = \{y \in L : p_j(x - y) < \varepsilon\}.$$ 

Describe these sets geometrically. What is the topology generated by the collection of semi-norms $\{p_1\}$? Is it Hausdorff? What is the topology generated by the collection of semi-norms $\{p_1, p_2\}$? Is it Hausdorff?

Solution:

For $x = (x_1, x_2)$, the set $B_{x,1,\varepsilon}$ is a vertical strip of width $2\varepsilon$ centered around $x_1$. The set $B_{x,2,\varepsilon}$ is a horizontal strip of width $2\varepsilon$ centered around $x_2$.

The topology $\mathcal{T}_1$ generated by $\{p_1\}$ is the topology on the real line. In other words, $\Omega \in \mathcal{T}_1$ iff $\Omega = \Omega_1 \times \mathbb{R}$ where $\Omega_1$ is an open set on the line. This topology is not Hausdorff. For a counter-example, set $x = (0, 0)$ and $y = (0, 1)$. Then if $\Omega \in \mathcal{T}_1$ we have

$$x \in \Omega \iff y \in \Omega.$$ 

As far as $\mathcal{T}_1$ is concerned, the points $x$ and $y$ are not distinct.

The topology generated by $\{p_1, p_2\}$ has as its base $\mathcal{B}$ intersections of open sets in $\mathcal{T}_1$ and $\mathcal{T}_2$. This means that $\mathcal{B}$ consists of all open rectangles in the plane. These generate the standard topology on $\mathbb{R}^2$, which is Hausdorff.