Problem 1: Consider the Hilbert space $H = l^2(\mathbb{N})$; let $e_n$ denote the canonical basis vectors. Which of the following sequences converge weakly? Which have convergent subsequences?

(a) $x_n = n e_n$.

(b) $y_n = n^{-1/2} \sum_{j=1}^n e_j$.

(c) $x_n = e_n + e_m$ where $m = 1 + \text{mod}(n, 2)$.

Problem 2: Consider the Hilbert space $H = L^2([-\pi, \pi])$, and the sequence of functions $\varphi_n(x) = x^2 \sin(nx)$. Does $(\varphi_n)_{n=1}^\infty$ converge strongly in $H$? Does $(\varphi_n)_{n=1}^\infty$ converge weakly in $H$? If you answer yes to either question, specify the limit.

Problem 3: Let $A$ denote a self-adjoint operator on a Hilbert space $H$. Let $u$ denote an element of $H$ and set $u_n = e^{i n A} u$. Prove that $(u_n)_{n=1}^\infty$ has a weakly convergent subsequence.

Problem 4: Let $H_1$ and $H_2$ be two Hilbert spaces. Let $U : H_1 \to H_2$ be a unitary operator, and let $A_1 \in \mathcal{B}(H_1)$ be a self-adjoint operator. Define the operator $A_2 \in \mathcal{B}(H_2)$ by $A_2 = U A_1 U^{-1}$. Prove that $A_2$ is self-adjoint.

Problem 5 (optional): Consider the Hilbert space $H = L^2([-\pi, \pi])$, and let $\mathcal{P}$ denote the set of trigonometric polynomials (which is dense in $H$). For $u \in \mathcal{P}$, let $A$ denote the operator $Au = 100 u + 18 u'' + u''''$. Prove that

$$\sup_{u \in \mathcal{P}, \|u\|=1} \langle Au, u \rangle = \infty.$$

Conclude that $A$ cannot be extended to a bounded linear operator on $H$. Prove that for $u, v \in \mathcal{P}$, $\langle Au, v \rangle = \langle A u, v \rangle$. Determine

$$\inf_{u \in \mathcal{P}, \|u\|=1} \langle Au, u \rangle.$$

Prove that

$$\langle u, v \rangle_A = \langle A u, v \rangle$$

is a bilinear form on $\mathcal{P}$. Prove that on $\mathcal{P}$, the norm $\| \cdot \|_A$ induced by $\langle \cdot, \cdot \rangle_A$ is equivalent to the norm

$$\|u\|_{H^2(\mathbb{T})} = \sqrt{\|u\|_{L^2(\mathbb{T})}^2 + \|u''\|_{L^2(\mathbb{T})}^2},$$

Conclude that the closure of $\mathcal{P}$ under the norm $\| \cdot \|_A$ is the space $H^2(\mathbb{T})$ (as defined in Section 7.2).