Homework set 3 — APPM5450, Spring 2017

From the textbook: 8.6. Optional: 8.5.

**Problem 1:** Let $H$ be a Hilbert space, and let $(\varphi_n)_{n=1}^\infty$ denote an orthonormal basis for $H$. Given a bounded sequence of complex number $(\lambda_n)_{n=1}^\infty$, define the operator $A$ by setting

$$Au = \sum_{n=1}^\infty \lambda_n \varphi_n \langle \varphi_n, u \rangle.$$  

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Prove that $A^*u = \sum_{n=1}^\infty \bar{\lambda}_n \varphi_n \langle \varphi_n, u \rangle$. Conclude that $A$ is self-adjoint iff all $\lambda_n$’s are real. When is $A$ skew-symmetric? When is $A$ non-negative / positive / coercive?

**Problem 2:** Consider the Hilbert space $H = L^2([-\pi, \pi])$, and the operator $A \in \mathcal{B}(H)$ defined by $[Au](x) = |x| u(x)$. Prove that $A$ is self-adjoint and positive, but not coercive. Prove that

$$\langle u, v \rangle_A = \langle Au, v \rangle$$

is an inner product on $H$, but that the topology generated by (the norm generated by) this inner product is not equivalent to the topology generated by the $L^2$-norm.

**Problem 3:** Set $H = \ell^2(\mathbb{Z})$ and let $R$ denote the right-shift operator (so that if $y = Rx$, then $y_n = x_{n-1}$). Construct $R^*$. Prove that $RR^* = R^*R = I$, which is to say that $R$ is “unitary.” (Is either the right or the left-shift operator on $\ell^2(\mathbb{N})$ unitary?)

**Problem 4:** Consider the Hilbert space $L^2(\mathbb{T})$. Let $k$ denote a continuous function on $\mathbb{T}^2$ that takes on complex values. Let $A$ denote the operator $[Au](x) = \int_{\mathbb{T}} k(x, y) u(y) dy$. Prove that $[A^*u](x) = \int_{\mathbb{T}} \overline{k(y, x)} u(y) dy$. Conclude that $A$ is self-adjoint iff $k(x, y) = \overline{k(y, x)} \forall x, y \in \mathbb{T}$. 
