Problem 7.1:

(a) Fix $\delta > 0$. For $x \in [-\delta/2, \delta/2]$ we have $1 + \cos x \geq 1 + \cos \frac{\delta}{2}$ so

$$
\frac{1}{c_n} = \int_{\mathbb{T}} (1 + \cos x)^n \, dx \geq \int_{-\delta/2}^{\delta/2} \left(1 + \cos \frac{\delta}{2}\right)^n \, dx = \delta \left(1 + \cos \frac{\delta}{2}\right)^n.
$$

Analogously, we find that

$$
\int_{|x|\geq \delta} c_n (1 + \cos x)^n \, dx \leq \int_{|x|\geq \delta} c_n (1 + \cos \delta)^n \, dx \leq c_n 2\pi (1 + \cos \delta)^n.
$$

Inserting (1) into (2) and taking the limit, we find (since $1 + \cos \delta < 1 + \cos(\delta/2)$)

$$
\lim_{n \to \infty} \int_{|x|\geq \delta} c_n (1 + \cos x)^n \, dx \leq \limsup_{n \to \infty} \frac{2\pi}{\delta} \left(1 + \cos \frac{\delta}{2}\right)^n = 0.
$$

(b) See lecture notes.

(c) No, since any function in $P$ is periodic. Consider for instance $f(x) = x$. Then for any $g \in P$

$$
||f - g||_u \geq \max(|f(0) - g(0)|, |f(2\pi) - g(2\pi)|) = \max(|g(0)|, |2\pi - g(0)|) \geq \pi.
$$

Problem 7.2: With $e_n(x) = e^{inx}/\sqrt{2\pi}$ we set

$$
f_N(x) = \sum_{n=-N}^{N} \alpha_n e_n(x), \quad \alpha_n = (e_n, f).
$$

Set $\beta = 1/\sqrt{2\pi}$. Then

$$
f_N(x) = \sum_{n=-N}^{N} \int_{-\pi}^{\pi} \beta e^{-iny} f(y) \, dy \beta e^{inx} = \int_{-\pi}^{\pi} \sum_{n=-N}^{N} \beta^2 e^{i(n-x-y)} f(y) \, dy.
$$

We will next simplify the kernel $D_N$. To this end, set $\alpha = e^{ix}$. Then

$$
D_N = \beta^2 \sum_{n=-N}^{N} \alpha^n.
$$

Moreover,

$$
\alpha D_N = \beta^2 \sum_{n=-N}^{N} \alpha^{n+1}.
$$

In consequence,

$$
(1 - \alpha) D_N = \beta^2 (\alpha^{-N} - \alpha^{N+1}).
$$

It follows that

$$
D_N = \beta^2 \frac{\alpha^{-N} - \alpha^{N+1}}{1 - \alpha} = \beta^2 \frac{\alpha^{-(N+1/2)} - \alpha^{N+1/2}}{\alpha^{-1/2} - \alpha^{1/2}} = \frac{1}{2\pi} \frac{\sin((N + 1/2)x)}{\sin(x/2)}.
$$

This proves part (a).
Next we set
\[ g_N = \frac{1}{N+1} \sum_{n=0}^{N} f_0 = \frac{1}{N+1} \sum_{n=0}^{N} D_n \ast f = \left( \frac{1}{N+1} \sum_{n=0}^{N} D_n \right) \ast f. \]

It remains to simplify \( F_N \). We have
\[
F_N = \frac{1}{N+1} \sum_{n=0}^{N} D_n = \frac{1}{N+1} \sum_{n=0}^{N} \beta^2 \alpha^{-n} - \alpha^{n+1} = \frac{\beta^2}{(N+1)(1-\alpha)^2} \left[ \alpha^{-(N+1)/2} - \frac{\alpha^{N+1}}{(N+1)(\alpha^{1/2} - \alpha^{1/2})^2} \right]
\]
\[
= \frac{\beta^2}{(N+1)(-2i \sin(x/2))^2} \left[ -2i \sin \left( \frac{(N+1)x}{2} \right) \right] = \frac{\beta^2}{(N+1)(\sin(x/2))^2} \left[ \sin \left( \frac{(N+1)x}{2} \right) \right].
\]
This proves part (b).

For (c), we observe that \( D_N \) takes on non-negative values, so it is not an approximate identity. Convolution by \( D_N \) provides the best approximation in the \( L^2 \)-norm, but it does not guarantee convergence in the uniform norm. In contrast, convolution by \( F_N \) does provide convergence in the uniform norm as long as \( f \in C(T) \).

**Problem 7.3:** Start by proving that the two putative bases are in fact orthonormal sets. Then it remains to prove that their closures span the set.

Fix an \( f \in L^2(J) \) with \( J = [0, \pi] \). To construct a sequence \( f_N \) such that \( \| f - f_N \|_{L^2(J)} \to 0 \), extend \( f \) to the function
\[
\tilde{f}(x) = \begin{cases} 
  f(x) & x \geq 0, \\
  -f(-x) & x < 0.
\end{cases}
\]
Then let \( f_N \) be the standard Fourier series of \( \tilde{f} \). Prove that the terms in this series are all sine functions. Since the exponentials form a basis, we know that \( \| \tilde{f} - f_N \|_{L^2(I)} \to 0 \) where \( I = [-\pi, \pi] \). Since \( \| \tilde{f} - f_N \|_{L^2(I)} = \sqrt{2} \| f - f_N \|_{L^2(J)} \), we then find that \( f_N \to f \) in \( L^2(J) \).

To prove that the cosines form a basis, repeat the argument, but do it with the symmetric continuation of \( f \) instead of the anti-symmetric one. In other words, set
\[
\tilde{f}(x) = \begin{cases} 
  f(x) & x \geq 0, \\
  f(-x) & x < 0,
\end{cases}
\]
and then use that the Fourier series for \( \tilde{f} \) involves only cosines.

**Problem 7.4:** This is a straight-forward calculation. You may want to look the correct answer up in a table to make sure you got the answer right.
Problem 7.5: The argument for the case $d = 1$ was done in class (see posted lecture notes). This argument can easily be modified to the case of $d$ dimensions. Let $f_N$ denote the partial Fourier sum. We need to prove that $(f_N)$ is Cauchy with respect to the uniform norm. If $M < N$, we find

$$|f_M(x) - f_N(x)| = \left| \sum_{M < |n| \leq N} \alpha_n e_n(x) \right| \leq \sum_{M < |n| \leq N} |\alpha_n|$$

$$\leq \left( \sum_{M < |n| \leq N} |n|^{-2k} \right)^{1/2} \left( \sum_{M < |n| \leq N} |n|^{2k} |\alpha_n|^2 \right)^{1/2}$$

$$\sim \left( \int_{M \leq |x| \leq N} |x|^{-2k} dx \right)^{1/2} \|f\|_{H^k} \sim \left( \int_{M}^{N} r^{-2k} r^{d-1} dr \right)^{1/2} \|f\|_{H^k}$$

$$\leq \left( \int_{M}^{\infty} r^{-2k} r^{d-1} dr \right)^{1/2} \|f\|_{H^k} = \frac{1}{\sqrt{2k - d} M^{k-d/2}} \|f\|_{H^k}.$$
**Problem 1:** Suppose that $H$ is a Hilbert space, and that $(\psi_n)_{n=1}^\infty$ is an ON-set in $H$. Let $\mathcal{P}$ denote the set of finite linear combinations of elements in $(\psi_n)_{n=1}^\infty$. Prove that:

$$(\psi_n)_{n=1}^\infty \text{ is a basis for } H \iff \mathcal{P} \text{ is dense in } H.$$

**Solution:** Suppose first that $(\psi_n)_{n=1}^\infty$ is a basis. Given any $f \in H$, define its partial expansion in $(\psi_n)$ as usual:

$$f_N = \sum_{n=1}^{N} (\psi_n, f) \psi_n \quad (3)$$

Since $(\psi_n)$ is a basis, we know that $f_N \to f$ in norm. Since $f_N \in \mathcal{P}$, this proves that any function can be approximated arbitrarily well by functions in $\mathcal{P}$.

Suppose next that $\mathcal{P}$ is dense. Fix an $f \in H$, and define its partial expansion $f_N$ as in (3). We need to prove that $f_N \to f$. Fix any $\varepsilon > 0$. Since $\mathcal{P}$ is dense, there is a $g \in \mathcal{P}$ such that $||f - g|| < \varepsilon$. Let $N$ be a number such that $g \in \text{Span}(\psi_1, \psi_2, \ldots, \psi_N) =: \mathcal{P}_N$. Now suppose that that $M \geq N$. Then since $g \in \mathcal{P}_M$, and $f_M$ is the best possible approximant within $\mathcal{P}_M$, we find

$$||f - f_M|| = ||f - g|| < \varepsilon.$$

This shows that $f_N \to f$.

**Problem 2:** Suppose that $f, g \in C(\mathbb{T})$. Prove that:

(a) $f * g \in C(\mathbb{T})$.

(b) $f * g = g * f$.

**Solution:**

(a) Set $h = f * g$. That $h$ is periodic follows directly from the periodicity of $f$:

$$h(x + 2\pi) = \int_\mathbb{T} f(x + 2\pi - y)g(y)dy = \int_\mathbb{T} f(x - y)g(y)dy = h(x).$$

Next we prove continuity. Fix $\varepsilon > 0$. Since $f$ is uniformly continuous, there is a $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon/(2\pi||g||)$ whenever $|x - x'| < \delta$. Now suppose that $|x - x'| < \delta$. Then

$$|h(x) - h(x')| = |\int_\mathbb{T} (f(x-y) - f(x'-y))g(y)dy| \leq \int_\mathbb{T} |f(x-y) - f(x'-y)||g(y)|dy \leq \int_\mathbb{T} \frac{\varepsilon}{2\pi||g||} \cdot ||g||dy = \varepsilon.$$

(b) Simply use the change of variables $z = x - y$ in the integral. You need to verify that the limits and the minus signs work out as they should, but that should not be hard.