Problem 1: (12p) Let $A$ be a self-adjoint bounded compact linear operator on a separable Hilbert space $H$. Which statements are necessarily true (no motivation required):

(a) $H$ has an ON-basis of eigenvectors of $A$.
   \textit{TRUE.} (Note that when $A$ has a null-space, you can just add an ON-basis for the null-space to the set of evcs associated with non-zero evals.)

(b) If $(e_n)_{n=1}^\infty$ is an ON-sequence, then $\lim_{n \to \infty} ||Ae_n|| = 0$.
   \textit{TRUE.} You know that $e_n \rightharpoonup 0$, and since $A$ is compact, it follows that $Ae_n \to 0$.

(c) For any $\lambda \in \mathbb{C}$, the subspace $\ker(A - \lambda I)$ is necessarily finite dimensional.
   \textit{FALSE.} If $\lambda = 0$, then the nullspace can be infinite dimensional.

(d) $\sigma_c(A) = \emptyset$.
   \textit{FALSE.} The origin can be in the continuum spectrum.

(e) $\sigma_r(A) = \emptyset$.
   \textit{TRUE.} Since $A$ is self-adjoint.

(f) $||A||$ is necessarily an eigenvalue of $A$.
   \textit{FALSE.} It is possible that only $-||A||$ is an eval.

Problem 2: (12p) Let $P$ be a projection on a Hilbert space $H$. Which of the following statements are necessarily correct (no motivation required):

(a) The spectral radius $r(P)$ is either precisely zero or precisely one.
   \textit{TRUE.} This follows from $r(P) = \lim_{n \to \infty} ||P^n||^{1/n}$ and $P^n = P$.

(b) $\sigma(P) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$.
   \textit{TRUE.} This follows from (a).

(c) $\sigma(P) \subseteq \mathbb{R}$.
   \textit{(This problem is harder than I had intended. No points were deducted.)}

(d) If $P$ is orthogonal, then $\sigma(P) \subseteq \{0, 1\}$.
   \textit{TRUE.} You know that $\sigma(P)$ is real, and that $P = I$ on its range.

(e) If $||Px|| = ||x||$ for every $x \in H$, then $P$ is necessarily the identity.
   \textit{TRUE.} Recall that $P = I$ on its range, and if $||Px|| = ||x||$ for every $x$, then $\ker(P) = \{0\}$.

(f) If there exist $x \in \text{ran}(P)$ and $y \in \ker(P)$ such that $\langle x, y \rangle \neq 0$, then $||P|| > 1$.
   \textit{TRUE.} See proof that $P$ is S-A iff $||P|| = 1$ or 0.

Problem 3: (25p) Let $H$ be a Hilbert space, and let $A$ be a bounded linear operator on $H$, so that $A \in \mathcal{B}(H)$.

(a) Define the resolvent set $\rho(A)$.

(b) Prove that $\rho(A)$ is an open set.

\textit{See course notes for solution.}
Problem 4: (25p) Define a map $T : S(\mathbb{R}) \to \mathbb{C}$ via
\[ T(\varphi) = \lim_{\varepsilon \searrow 0} \left( \int_{-\infty}^{-\varepsilon} \frac{1}{x} \varphi(x) \, dx + \int_{\varepsilon}^{\infty} \frac{1}{x} \varphi(x) \, dx \right). \]

Prove that $T$ is a continuous functional on $S$. (You do not need to prove linearity.) What can you say about the order of $T$?

Note: Recall that the order of a distribution is the lowest number $m$ for which a bound of the form $|T(\varphi)| \leq C \sum_{\ell \leq k} \sum_{|\alpha| \leq m} ||\varphi||_{\ell, \alpha}$ holds.

Solution: Set $\psi(x) = \varphi(x) - \varphi(-x)$.

For $x > 0$, we find that $|\psi(x)| = \frac{1}{x} \int_{-x}^{x} \varphi'(y) \, dy \leq 2||\varphi||_{1,0}$.

For $x > 0$, we also find that $|\psi(x)| = |x|^{-1}(1 + x^2)^{-1/2}(1 + x^2)^{1/2}|\varphi(x) + \varphi(-x)| \leq \frac{1}{x^2} 2||\varphi||_{0,1}$.

Via a change of variable, we find $T(\varphi) = \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\infty} \psi(x) \, dx$. Note that $\psi$ is a continuous bounded function, so the limit exists and $T(\varphi) = \int_{0}^{\infty} \psi(x) \, dx$. Then
\[ |T(\varphi)| = \left| \int_{0}^{1} \psi(x) \, dx + \int_{1}^{\infty} \psi(x) \, dx \right| \leq \int_{0}^{1} 2||\varphi||_{1,0} \, dx + \int_{1}^{\infty} \frac{1}{x^2} 2||\varphi||_{0,1} \, dx = 2||\varphi||_{1,0} + 2||\varphi||_{0,1}. \]

This proves that $T$ has order at most 1. Two points were deducted if you omit the “at most” part.

Full credit was awarded without providing a proof that the order cannot be 0. But for completeness, this part of the arguments can be done as follows: For $n$ positive, pick $\varphi_n \in S$ such that

- $|\varphi_n(x)| \leq 1$ for all $x$.
- $|\varphi_n(x)| = 0$ for all $x$ such that $|x| \geq 2$.
- $\varphi_n(x) \geq 0$ for $x \geq 0$.
- $\varphi_n(x) \leq 0$ for $x \leq 0$.
- $\varphi_n(x) = 1$ for $x \in [1/n, 1]$.
- $\varphi_n(x) = -1$ for $x \in [-1, -1/n]$.

Observe that then $||\varphi_n||_{0,k} \leq 5^{k/2}$ for every $n$, but
\[ T(\varphi_n) \geq \int_{1/n \leq |x| \leq 1} \frac{1}{x} \varphi(x) \, dx = \int_{1/n \leq |x| \leq 1} \frac{1}{x} \, dx = 2 \int_{1/n}^{1} \frac{1}{x} \, dx = 2 \log(n) \to \infty. \]

Incidentally, observe that for this sequence, we must necessarily have $||\varphi_n||_{1,0} = ||\varphi'_n||_{0,1} \geq n$, since $\varphi_n$ changes from the value -1 to value 1 in the distance $2/n$. 
**Problem 5:** (24p) Consider the Hilbert space $H = L^2(\mathbb{R})$. For this problem, we define $H$ as the closure of the set of all compactly supported smooth functions on $\mathbb{R}$ under the norm  
$$ ||u|| = \left( \int_{-\infty}^{\infty} |u(x)|^2 \, dx \right)^{1/2}. $$ 
Which of the following sequences converge weakly in $H$? Motive your answers briefly.

(a) $(u_n)_{n=1}^\infty$ where $u_n(x) = \begin{cases} 1 - |x - n|, & \text{for } x \in [n - 1, n + 1], \\ 0, & \text{for } x \in (-\infty, n - 1) \cup (n + 1, \infty). \end{cases}$

(b) $(v_n)_{n=1}^\infty$ where $v_n(x) = \sin(nx) e^{-x^2}$.

(c) $(w_n)_{n=1}^\infty$ where $w_n(x) = \begin{cases} 1 - |x/n - 1|, & \text{for } x \in [0, 2n] \\ 0, & \text{for } x \in (-\infty, 0) \cup (2n, \infty). \end{cases}$

**Solution:** Let $\Omega$ denote the set of smooth functions with compact support. These are by definition dense in $H$. We use the theorem that says that a sequence is weakly convergent iff it is bounded, and you have weak convergence when measured against any member in a dense set. In the solution, we use $\Omega$ as the dense set.

First observe that $(u_n)$ and $(v_n)$ are bounded, but $(w_n)$ is not. We can immediately rule out $(w_n)$.

Fix $f \in \Omega$. Set $M = \sup\{|x| : f(x) \neq 0\}$. Since $f$ has compact support, $M$ is bounded. Now if $n > M + 1$, we find that $(u_n, f) = 0$, so obviously $\lim_{n \to \infty} (u_n, f) = 0$. This shows $u_n \to 0$.

Again fix $f \in \Omega$, and set $g(x) = e^{-x^2}f(x)$. Then as $n \to \infty$,  
$$ |(v_n, f)| = \left| \int_{-\infty}^{\infty} \sin(nx) g(x) \, dx \right| = \left| -\frac{1}{n} \int_{-\infty}^{\infty} \cos(nx) g'(x) \, dx \right| \leq \frac{1}{n} \int_{-\infty}^{\infty} |g'(x)| \, dx \to 0. $$

We use that $g$ has compact support so the boundary terms in the partial integration vanish, and $\int |g'| < \infty$. This shows $v_n \to 0$.

In summary, $(u_n)$ and $(v_n)$ both converge weakly to zero, but $(w_n)$ does not converge weakly.