Problem 12.8: We want to prove that
\[ ||f - f_n||_p^p = \int |f - f_n|^p \to \infty. \]
We know that \( |f - f_n|^p \to 0 \) pointwise, so if we can only justify moving the limit inside the integral, we’ll be done.

First note that
\[ |f(x)| = \lim_{n \to \infty} |f_n(x)| \leq |g(x)|. \]
Then we can dominate the integrand as follows:
\[ |f - f_n|^p \leq (|f| + |f_n|)^p \leq (|g| + |g|)^p \leq 2^p |g|^p. \]
Since \( \int |g|^p < \infty \), we find that the Lebesque dominated convergence theorem applies, and so
\[ \lim_{n \to \infty} |f - f_n|^p = \lim_{n \to \infty} \int |f - f_n|^p = \{\text{LDCT}\} = \int (\lim_{n \to \infty} |f - f_n|^p) = \int 0 = 0. \]

Problem 12.16: Fix \( f \in L^p \) and \( \varepsilon > 0 \). We want to prove that there exists a \( \delta > 0 \) such that for \( |h| < \delta \), we have \( ||f - \tau_h f||_p < \varepsilon \).

First pick \( \varphi \in C_c \) such that \( ||f - \varphi||_p < \varepsilon/3 \). Then
\[ ||f - \tau_h f||_p = ||f - \varphi||_p + ||\varphi - \tau_h \varphi||_p + ||\tau_h \varphi - \tau_h f||_p \]
\[ = ||f - \varphi||_p + ||\varphi - \tau_h \varphi||_p + ||\varphi - f||_p < \varepsilon/3 + ||\varphi - \tau_h \varphi||_p + \varepsilon/3. \]
Set \( R = \sup \{|x| : \varphi(x) \neq 0\} \). Since \( \varphi \) is uniformly continuous, there exists a \( \delta \) such that if \( |x - y| < \delta \), then \( |\varphi(x) - \varphi(y)| < \varepsilon/(3\mu(B_{R+1}(0))^{1/p}). \) Then, if \( h < \min(\delta, 1), \)
\[ ||\varphi - \tau_h \varphi||_p = \int_{B_{R+1}(0)} |\varphi(x) - \varphi(x - h)|^p dx \leq \int_{B_{R+1}(0)} \frac{\varepsilon^p}{3^p \mu(B_{R+1}(0))} dx < \frac{\varepsilon^p}{3^p}. \]

Problem 12.17: For \( n = 1, 2, 3, \ldots \), set \( I_n = (2^{-n}, 2^{-n+1}) \), and \( f_n = 2^{n/p} \chi_{I_n}. \) Then \( ||f_n||_p = 1 \) for all \( n \). Suppose \( m \neq n \), then
\[ ||f_n - f_m||_\infty = 1, \]
and for \( p \in [1, \infty) \) we have
\[ ||f_n - f_m||_p = \left( \int_0^1 \left( 2^n \chi_{I_n} + 2^m \chi_{I_m} \right)^p dx \right)^{1/p} = 2^{1/p}. \]
No subsequence of \( (f_n)_{n=1}^\infty \) can be Cauchy, and therefore no subsequence can converge.

Problem 12.18: For \( n = 1, 2, 3, \ldots \), set \( I_n = (2^{-n}, 2^{-n+1}) \), and \( f_n = 2^n \chi_{I_n}. \) Let \( (f_n)_{j=1}^\infty \) be a subsequence of \( (f_n)_{n=1}^\infty \). Define \( g \in L^\infty \) by
\[ g = \sum_{j=1}^\infty (-1)^j \chi_{I_{n_j}}, \]
and define \( \varphi \in (L^1)^* \) via \( \varphi(f) = \int fg \). Then \( \varphi(f_{n_j}) = (-1)^j \) (verify!) and so \( (f_{n_j}) \) cannot converge weakly. Since \( L^1 \) is not reflexive, this does not contradict that Banach-Alaoglu theorem.
Problem 12.13: Set $I = [0, 1]$ and let $\Omega$ be a dense set in $L^\infty(I)$. For $r \in I$, set $f_r = \chi_{[0, r]}$, and pick $x_r \in \Omega \cap B_{1/3}(f_r)$. Since $||f_r - f_s|| = 1$ if $s \neq r$, we find that $||x_r - x_s|| \geq ||f_r - f_s|| - ||f_r - x_r|| - ||f_s - x_s|| \geq 1/3$, so all the $x_r$’s are distinct. Therefore, $\Omega$ must be uncountable, and $L^\infty$ cannot be separable.

To prove that $C(I)$ cannot be dense in $L^\infty(I)$, simply note that if $f = \chi_{[0, 1/2]}$, and $\varphi \in C(I)$, then

$$||f - \varphi||_{\infty} \geq \max(||\varphi(1/2)||, |1 - \varphi(1/2)|) \geq 1/2$$

(verify this!).

An alternative argument for why $C(I)$ cannot be dense in $L^\infty(I)$: If $\varphi_n \in C(I)$, and $\varphi_n \to f$ in the supnorm, then $(\varphi_n)$ is a Cauchy sequence with respect to the uniform norm (when applied to continuous functions, the uniform norm and the $L^\infty$ norms are identical). Therefore, there exists a continuous function $\varphi$ such that $\varphi_n \to \varphi$ uniformly. Then $f(x) = \varphi(x)$ almost everywhere. But not every equivalence class function in $L^\infty$ has a continuous function in it (for instance $f = \chi_{[0, 1/2]}$).

Problem 12.14: Let $p$ and $q$ be such that $1 \leq p < q \leq \infty$.

First we construct a function $f \in L^p \setminus L^q$. Let $\alpha$ be a non-negative number and set $f(x) = x^{-\alpha} \chi_{[0, 1]}$. Then

$$||f||^p_p = \int_0^1 x^{-\alpha p} \, dx,$$

which is finite if $\alpha p < 1$. Moreover

$$||f||^q_q = \int_0^1 x^{-\alpha q} \, dx$$

which is infinite if $\alpha q > 1$. Consequently, $f \in L^p \setminus L^q$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

To construct a function $f \in L^q \setminus L^p$, set $f = x^{-\alpha} \chi_{[1, \infty)}$. Then

$$||f||^p_p = \int_1^\infty x^{-\alpha p} \, dx$$

which is infinite if $\alpha p < 1$. Moreover

$$||f||^q_q = \int_1^\infty x^{-\alpha q} \, dx$$

which is finite if $\alpha q > 1$. Thus, $f \in L^1 \setminus L^p$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

(The arguments above need slight modifications if $q = \infty$, but the idea is the same.)

Consider the function

$$f(x) = \frac{1}{(|x| (1 + \log^2 |x|))^{1/2}}.$$
That $f \in L^2$ is clear, since
\[
\|f\|^2 L^2 = \int_{-\infty}^{\infty} \frac{1}{|x|(1 + \log^2 |x|)} \, dx = 2 \int_0^{\infty} \frac{1}{x(1 + \log^2 x)} \, dx = \{x = e^t\} \\
2 \int_{-\infty}^{\infty} \frac{1}{e^t(1 + t^2)} e^t \, dt = 2\pi.
\]
Moreover, if $p > 2$, then note that there exists a $\delta > 0$ such that
\[
x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1
\]
when $x \in (0, \delta)$. Then
\[
\|f\|^p p \geq \int_0^{\delta} \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} \, dx = \int_0^{\delta} \frac{1}{x} \, \underbrace{x^{(p-2)/2} (1 + \log^2 x)^{p/2}}_{\geq 1} \, dx = \infty.
\]
Analogously, if $p < 2$, then there exists an $M$ such that
\[
x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1
\]
when $x \geq M$. Then
\[
\|f\|^p p \geq \int_M^{\infty} \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} \, dx = \int_M^{\infty} \frac{1}{x} \, \underbrace{x^{(p-2)/2} (1 + \log^2 x)^{p/2}}_{\geq 1} \, dx = \infty.
\]

**Problem 12.15:** Let $\alpha \in (0, 1)$, and let $m, n \in (1, \infty)$ be such that $1/m + 1/n = 1$ (we will determine suitable values for $\alpha, m, n$ later). Then from Hölder’s inequality we obtain
\[
\|f\|_r \leq \left(\left\|f\|_p^p\right\| f^{|1-\alpha|} \right)^{1/m} \left(\left\|f\|_q^q\right\| f^{|1-\alpha|} \right)^{1/n}.
\]
In order to obtain the desired right hand side, we must pick $\alpha, m, n$ so that
\[
\alpha m = p, \\
(1-\alpha)n = q, \\
(1/m) + (1/n) = 1.
\]
To obtain an equation for $\alpha$, we eliminate $m$ and $n$:
\[
\frac{(1-\alpha)r}{q} = \frac{1}{n} = 1 - \frac{1}{m} = 1 - \frac{\alpha r}{p}.
\]
Solving for $\alpha$ we obtain
\[
\alpha = \frac{pq - pr}{rq - rp} = \frac{1/r - 1/q}{1/p - 1/q}.
\]
Equation (1) now takes the form
\[
\|f\|_r \leq \left(\left\|f\|_p^{p/m} \left\|f\|_q^{q/n}\right\| f^{|1-\alpha|} \right)^{1/r} = \|f\|^p_{p/m} \|f\|^q_{q/n}.
\]
Finally note that
\[
\frac{p}{mr} = \alpha = \frac{1/r - 1/q}{1/p - 1/q}, \\
\frac{q}{nr} = 1 - \alpha = 1 - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/r}{1/p - 1/q}.
\]
**Problem 1:** Let $\lambda$ be a real number such that $\lambda \in (0, 1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that

\[ a^\lambda \ b^{1-\lambda} \leq \lambda \ a + (1 - \lambda) \ b, \]

with equality iff $a = b$.

**Solution:** For $b = 0$ equation (2) reduces to $0 \leq \lambda a$ which is clearly true.

When $b \neq 0$ we divide (2) by $b$ and set $t = a/b$ to obtain

\[ t^\lambda \leq \lambda t + 1 - \lambda. \]

Set

\[ f(t) = \lambda t + 1 - \lambda - t^\lambda. \]

We need to prove that $f(t) \geq 0$ when $t \geq 0$. First note that $f(0) = 1 - \lambda > 0$ and that $\lim_{t \rightarrow \infty} f(t) = \infty$. Since $f$ is differentiable, we therefore need only investigate the points where $f'(t) = 0$. We find

\[ f'(t) = \lambda - \lambda t^\lambda - 1 \]

so $f'(t) = 0$ happens only when $t = 1$. Now $f(1) = 0$ so it follows that $f(t) \geq 0$ for all $t \geq 0$, and that $f(t) = 0$ iff $t = 1$ (which is to say $a = b$).

**Problem 2:** [Hölder’s inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $(X, \mu)$ be a measure space, and suppose that $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that

\[ ||fg||_1 \leq ||f||_p ||g||_q. \]

Prove that equality holds iff $\alpha |f|^p = \beta |g|^q$ for some $\alpha, \beta$ such that $\alpha \beta \neq 1$.

**Solution:** Suppose $||f||_p = 0$, then $f = 0$ a.e. and so (3) holds since both sides are identically zero. Analogously, (3) holds when $||g||_q = 0$.

Now suppose $||f||_p \neq 0$ and $||g||_q \neq 0$. Set

\[ a = \left| \frac{f(x)}{||f||_p} \right|^p, \quad b = \left| \frac{g(x)}{||g||_q} \right|^q, \quad \lambda = \frac{1}{p}. \]

Then invoke (2), observing that $q(1 - \lambda) = q(1 - 1/p) = q(1/q) = 1$, to obtain

\[ \frac{|f(x)| \ |g(x)|}{||f||_p \ ||g||_q} \leq \frac{1}{p} \ |f(x)|^p + \left( 1 - \frac{1}{p} \right) \ |g(x)|^q. \]

Integrate over $X$ to obtain

\[ \frac{1}{||f||_p \ ||g||_q} \int_X |f(x)| \ |g(x)| \ d\mu(x) \leq \frac{1}{p} \frac{||f||_p^p}{||f||_p^p} + \left( 1 - \frac{1}{p} \right) \frac{||g||_q^q}{||g||_q^q}. \]

Multiply by $||f||_p \ ||g||_q$ to obtain (3).
**Problem 3:** [Minkowski’s inequality] Let \((X, \mu)\) be a measure space, and let \(p\) be a real number such that \(1 \leq p \leq \infty\). Prove that for \(f, g \in L^p(X, \mu)\),
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

**Solution:**

Suppose \(p = 1\):
\[
\|f + g\|_1 = \int |f(x) + g(x)| \leq \int (|f(x)| + |g(x)|) = \int |f(x)| + \int |g(x)| = \|f\|_1 + \|g\|_1.
\]

Suppose \(p = \infty\):
\[
\|f + g\|_\infty = \text{ess sup} |f(x) + g(x)| \leq \text{ess sup}(|f(x)| + |g(x)|)
\]
\[
\leq \text{ess sup} |f(x)| + \text{ess sup} |g(x)| = \|f\|_\infty + \|g\|_\infty.
\]

Suppose \(p \in (1, \infty)\): The triangle inequality yields
\[
|f(x) + g(x)|^p = |f(x) + g(x)| |f(x) + g(x)|^{p-1} \leq (|f(x)| + |g(x)|) |f(x) + g(x)|^{p-1}.
\]
Integrate both sides:
\[
\|f + g\|_p^p \leq \int |f(x)| |f(x) + g(x)|^{p-1} + \int |g(x)| |f(x) + g(x)|^{p-1}.
\]
Now apply Hölder:
\[
\|f + g\|_p^p \leq \|f\|_p \|f + g|^{p-1}\|_q + \|g\|_p \|f + g|^{p-1}\|_q = (\|f\|_p + \|g\|_p) \left( \int |f(x) + g(x)|^{q(p-1)} \right)^{1/q}.
\]
Now use that \(q = 1/(1 - 1/p) = p/(p - 1)\) to see that \(q(p - 1) = p\) to get
\[
\|f + g\|_p^p \leq \|f\|_p \|f + g|^{p-1}\|_q + \|g\|_p \|f + g|^{p-1}\|_q = (\|f\|_p + \|g\|_p) \left( \int |f(x) + g(x)|^p \right)^{1/q} = (\|f\|_p + \|g\|_p) \|f + g\|_p^{p/q}.
\]
Observe that \(p/q = p(1 - 1/p) = p - 1\) to obtain
\[
\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}
\]
which gives Minkowski upon division by \(\|f + g\|_p^{p-1}\).