Homework set 14 — APMM5450, Spring 2014


**Problem 1:** Let $\lambda$ be a real number such that $\lambda \in (0,1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that
\[
a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b,
\]
with equality iff $a = b$.

*Hint:* Consider the case $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.

**Problem 2:** [Hölder’s inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $(X, \mu)$ be a measure space, and suppose that $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that
\[
\|fg\|_1 \leq \|f\|_p \|g\|_q.
\]
Prove that equality holds iff $\alpha|f|^p = \beta|g|^q$ for some $\alpha, \beta$ such that $\alpha \beta \neq 1$.

*Hint:* Consider first the case where $\|f\|_p = 0$ or $\|g\|_q = 0$. For the case $\|f\|_p \|g\|_q \neq 0$, use (1) with
\[
a = \frac{|f(x)|^p}{\|f\|_p}, \quad b = \frac{|g(x)|^q}{\|g\|_q}, \quad \lambda = \frac{1}{p}.
\]

**Problem 3:** [Minkowski’s inequality] Let $(X, \mu)$ be a measure space, and let $p$ be a real number such that $1 \leq p \leq \infty$. Prove that for $f, g \in L^p(X, \mu)$,
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

*Hint:* Consider the cases $p = 1, \infty$ separately. For $p \in (1, \infty)$, note that
\[
|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|) |f(x) + g(x)|^{p-1}, \quad \forall x \in X.
\]
Then integrate both sides of (3) and apply (2) to the right hand side.