Problem 9.21: Suppose $A \in \mathcal{B}(H)$ is such that
$$\text{Re}(x, Ax) \leq 2\alpha \|x\|^2.$$ 
Prove that the solution $x = x(t)$ of $x'(t) = Ax(t)$ satisfies 
$$\|x(t)\| \leq e^{\alpha t} \|x(0)\|.$$

Note: The book may have a typo — the bound seems off by a factor of two. Consider for instance $Ax = 2\alpha x$, then 
$x(t) = e^{2\alpha t}x(0)$.

Solution: Set $f(t) = \|x(t)\|^2$.
Then 
$$f'(t) = \frac{d}{dt}(x, x) = (x', x) + (x, x') = (Ax, x) + (x, Ax) = 2\text{Re}(x, Ax) \leq 4\alpha \|x(t)\|^2 = 4\alpha f(t).$$
By the Grönwall inequality, we find 
$$\|x(t)\|^2 = f(t) \leq f(0) \exp(\int_0^t 4\alpha \, ds) = f(0) e^{4\alpha t} = \|x(0)\|^2 e^{4\alpha t}.$$ 
Extract the square root to obtain the desired bound.

Problem 9.22: Let $A$ be compact and non-negative. Prove that there exists a unique compact non-negative operator $B$ such that $B^2 = A$.

Solution: Since $A$ is self-adjoint and compact, there is an ON-basis $(\varphi_n)_{n=1}^\infty$ of eigen-vectors of $A$. 
$A\varphi_n = \lambda_n \varphi_n$. We know $|\lambda_n| \to 0$ since $A$ is compact, and $\lambda_n \geq 0$ since $A$ is non-negative.

Existence: Set $B = \sum_{n=1}^\infty \sqrt{\lambda_n} P_n$ where $P_n x = (\varphi_n, x) \varphi_n$. It is easily shown that $B^2 = A$ and that $B$ is compact and non-negative.

Observe that from the construction of $B$, it follows that if $\psi$ is a vector such that $A \psi = \lambda \psi$, then $B \psi = \sqrt{\lambda} \psi$.

Uniqueness: Suppose that $C$ is a non-negative compact operator such that $C^2 = A$. We need to show that $C = B$, where $B$ is the operator constructed above. Since $C$ is compact and self-adjoint, there is an ON-basis $(\psi_n)_{n=1}^\infty$ such that $C \psi_n = \mu_n \psi_n$. Now observe that 
$$A \psi_n = C^2 \psi_n = C (\mu_n \psi_n) = \mu_n^2 \psi_n$$
so $\psi_n$ is an eigenvector of $A$ with eigenvalue $\mu_n^2$. It follows that 
$$B \psi_n = \sqrt{\mu_n^2} \psi_n = \mu_n \psi_n = C \psi_n.$$ 
(We know that $\sqrt{\mu_n^2} = \mu_n$ since $C$ must be non-negative, which implies that $\mu_n \geq 0$.)
Problem 1: Consider the Hilbert space $H = \mathbb{C}^n$. Let $A \in \mathcal{B}(H)$, let $(e_j^n)_{j=1}^n$ be the canonical basis, and let $A$ have the representation

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

in the canonical basis. We define the Hilbert-Schmidt norm of $A$ as

$$||A||_{HS} = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}.$$

(a) Let $(\varphi(j))_{j=1}^n$ be any ON-basis for $H$. Show that $||A||_{HS}^2 = \sum_{j=1}^n ||A\varphi(j)||^2$.

(b) Show that $||A|| \leq ||A||_{HS} \leq \sqrt{n}||A||$ for any $A \in \mathcal{B}(H)$.

(c) Find $G, H \in \mathcal{B}(H)$ such that $||G||_{HS} = ||G||$ and $||H||_{HS} = \sqrt{n}||H||$.

Solution:

(a) Let $r^{(i)}$ denote the $i$'th row of $A$. Then

$$\sum_{j=1}^n ||A\varphi(j)||^2 = \sum_{j=1}^n \sum_{i=1}^n ||(r^{(i)}, \varphi(j))||^2 = \{\text{Parseval}\} = \sum_{i=1}^n ||r^{(i)}||^2 = ||A||_{HS}^2.$$

(b) For any $x$ a simply application of Cauchy-Schwartz yields

$$||Ax||^2 = \sum_{i=1}^n ||(r^{(i)}, x)||^2 \leq \sum_{i=1}^n ||r^{(i)}||^2 ||x||^2 = ||A||_{HS}^2 ||x||^2.$$

It follows that $||A|| \leq ||A||_{HS}$. Next, let $i$ be such that $||r^{(i)}|| = \max_j ||r^{(j)}||$. Then

$$||A||_{HS}^2 = \sum_{j=1}^n ||r^{(j)}||^2 \leq n ||r^{(i)}||^2 = n ||A^* e_i||^2 \leq n ||A^*|| = n ||A||,$$

where $e_i$ denotes the $i$'th canonical basis vector.

(c) For instance, let $G$ be the matrix consisting of all ones, and let $H$ be the identity matrix.
**Problem 2:** Let $H$ be a separable Hilbert space, and let $A \in \mathcal{B}(H)$. Suppose that $H$ has an ON-basis $(\varphi(j))_{j=1}^\infty$ such that
\[ \sum_{j=1}^\infty ||A\varphi(j)||^2 < \infty. \]
Prove that if $(\psi(j))_{j=1}^\infty$ is any other ON-basis, then
\[ \sum_{j=1}^\infty ||A\varphi(j)||^2 = \sum_{j=1}^\infty ||A\psi(j)||^2. \]

**Solution:** Set
\[ \alpha_{ji} = (A\varphi(j), \psi(i)) = (\varphi(j), A^*\psi(i)) \]
and
\[ \beta_{ik} = (A^*\psi(i), \psi(k)) = (\psi(i), A\psi(k)). \]
The proof consists of four applications of Parseval:
\[ \sum_{j=1}^\infty ||A\varphi(j)||^2 = \sum_{j=1}^\infty \sum_{i=1}^\infty |\alpha_{ji}|^2 = \sum_{i=1}^\infty ||A^*\psi(i)||^2 = \sum_{i=1}^\infty \sum_{k=1}^\infty |\beta_{ik}|^2 = \sum_{k=1}^\infty ||A\psi(k)||^2. \]
Note that the interchanges of summation order are permissible as all terms are non-negative.
**Problem 3:** Consider the linear space \( L = \mathbb{R}^2 \). Define for \( x = (x_1, x_2) \in L \) the seminorms
\[
p_1(x) = |x_1|, \quad p_2(x) = |x_2|.
\]
Construct for \( x \in L, \ j \in \{1, 2\}, \ \text{and} \ \varepsilon \in (0, \infty) \), the sets
\[
B_{x,j,\varepsilon} = \{ y \in L : p_j(y) - p_j(x) < \varepsilon \}.
\]
Describe these sets geometrically. What is the topology generated by the collection of semi-norms \( \{p_1\} \)? Is it Hausdorff? What is the topology generated by the collection of semi-norms \( \{p_1, p_2\} \)? Is it Hausdorff?

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**Solution:**

For \( x = (x_1, x_2) \), the set \( B_{x,1,\varepsilon} \) is a vertical strip of width \( 2\varepsilon \) centered around \( x_1 \). The set \( B_{x,2,\varepsilon} \) is a horizontal strip of width \( 2\varepsilon \) centered around \( x_2 \).

The topology \( T_1 \) generated by \( \{p_1\} \) is the topology on the real line. In other words, \( \Omega \in T_1 \) iff \( \Omega = \Omega_1 \times \mathbb{R} \) where \( \Omega_1 \) is an open set on the line. This topology is not Hausdorff. For a counter-example, set \( x = (0, 0) \) and \( y = (0, 1) \). Then if \( \Omega \in T_1 \) we have
\[
x \in \Omega \quad \Leftrightarrow \quad y \in \Omega.
\]

As far as \( T_1 \) is concerned, the points \( x \) and \( y \) are not distinct.

The topology generated by \( \{p_1, p_2\} \) has as its base \( \mathcal{B} \) intersections of open sets in \( T_1 \) and \( T_2 \). This means that \( \mathcal{B} \) consists of all open rectangles in the plane. These generate the standard topology on \( \mathbb{R}^2 \), which is Hausdorff.