9.10 (slightly reworded): Consider the Hilbert space $H = l^2(\mathbb{N})$ and the operators $L$ and $R$ defined by

$$
R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots),
$$

$$
L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).
$$

Determine and classify the spectra of $L$ and $R$.

Solution: First we determine the eigenvalues. Let $\lambda$ be a complex number and consider the equation $Rx = \lambda x$. It is easily seen that the only solution is $x = 0$ (ones needs to distinguish between the cases $\lambda = 0$ and $\lambda \neq 0$), and so $\sigma_p(R) = \emptyset$. Next consider the equation $Lx = \lambda x$. The only solution is $x = x_1(1, \lambda, \lambda^2, \ldots)$, so we can construct eigenvectors in $H$ if and only if $|\lambda| < 1$. Thus $\sigma_p(L) = \{\lambda : |\lambda| < 1\}$.

Consider $\lambda$ such that $|\lambda| > 1$: Since $||L|| = ||R|| = 1$, we immediately find that $\lambda \notin \sigma(L)$ and $\lambda \notin \sigma(R)$.

Consider $\lambda$ such that $|\lambda| < 1$: We already determined that $\lambda \in \sigma_p(L)$. Since $R = L^*$, this fact gives us information about the spectrum of $R$ as well since

$$
\text{ran}(R - \lambda I) = (\ker(R^* - \bar{\lambda} I))^\perp = (\ker(L - \bar{\lambda} I))^\perp \neq \{0\}.
$$

It follows that $\lambda \in \sigma(R)$, but that $\lambda \notin \sigma_c(R)$. Moreover, determined previously that $\sigma_p(R) = \emptyset$, so we must have that $\lambda \in \sigma_c(R)$.

Consider $\lambda$ such that $|\lambda| = 1$: First note that since the spectrum of any operator is closed, and the open unit disc belongs to the spectra of both $L$ and $R$, we know that $\lambda$ belongs to both $\sigma(L)$ and $\sigma(R)$. We also know that $\lambda$ is not an eigenvalue of either $L$ or $R$. Suppose that $\lambda$ is in the residual spectrum of $L$. It would then follow that $\bar{\lambda} \in \sigma_p(L^*) = \sigma_p(R)$, but we know that this is not the case. Thus $\lambda \in \sigma_c(L)$. The proof that $\lambda \in \sigma_c(R)$ is analogous.