Problem 12.2: We use \( \emptyset \) to denote disjoint unions.

(a) Suppose that \( A, B \in \mathcal{A} \). Then note that \( A \setminus B = A \cap B^c = (A^c \cup B)^c \). It now follows directly from the axioms that \( A \setminus B \in \mathcal{A} \).

(b) Set \( C = B \setminus A \). Then \( B = A \cup C \), so
\[
\mu(B) = \mu(A \cup C) = \mu(A) + \mu(C) \geq \mu(A).
\]

(c) Set \( C = A \cap B \). Then \( A = (A \setminus B) \cup C \) and \( B = (B \setminus A) \cup C \) so
\[
\mu(A \cup B) = \mu((A \setminus B) \cup C \cup (B \setminus A)) = \mu(A \setminus B) + \mu(C) + \mu(B \setminus A)
\leq \mu(A \setminus B) + \mu(C) + \mu(C) + \mu(B \setminus A)
= \mu((A \setminus B) \cup C) + \mu(C \cup (B \setminus A)) = \mu(A) + \mu(B).
\]

Problem 12.3: The trick is to write \( \bigcup_{n=1}^{\infty} A_n \) as a disjoint union. For \( n = 1, 2, 3, \ldots \) set \( B_n = A_{n+1} \setminus A_n \). Then
\[
\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left( \bigcup_{n=1}^{\infty} B_n \right),
\]
where there union on the right is a disjoint one. Now use additivity twice:
\[
\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \mu \left( A_1 \cup \left( \bigcup_{n=1}^{\infty} B_n \right) \right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n)
\]
\[
= \lim_{N \to \infty} \left( \mu(A_1) + \sum_{n=1}^{N} \mu(B_n) \right) = \lim_{N \to \infty} \mu \left( A_1 \cup \left( \bigcup_{n=1}^{N} B_n \right) \right) = \lim_{N \to \infty} \mu(A_N).
\]

For the second part, set \( C = \bigcap_{n=1}^{\infty} A_n \) and \( C_n = A_n \setminus A_{n+1} \). Then
\[
\mu(A_N) = \mu \left( C \cup \left( \bigcup_{n=N}^{\infty} C_n \right) \right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).
\]

Since \( \infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n) \), we find that
\[
\lim_{N \to \infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,
\]
which completes the proof. For the counterexample, consider \( X = \mathbb{R}^2 \), and \( A_n = \{ x = (x_1, x_2) : |x_2| < 1/n \} \). Then \( \mu(A_n) = \infty \) for all \( n \), but \( \bigcap_{n=1}^{\infty} A_n \) is the \( x_1 \)-axis, which has measure zero.

Problem 12.5: Straight-forward.
Problem 12.7:

*Reflexivity:* It is obvious that $f(x) = f(x)$ a.e.

*Symmetry:* If $f(x) = g(x)$ a.e., then obviously $g(x) = f(x)$ a.e.

*Transitivity:* Suppose that $f(x) = g(x)$ a.e. and that $g(x) = h(x)$ a.e. Set

- $A = \{ x : f(x) \neq g(x) \}$
- $B = \{ x : g(x) \neq h(x) \}$
- $C = \{ x : f(x) \neq h(x) \}$.

We know that $\mu(A) = \mu(B) = 0$, and we want to prove that $\mu(C) = 0$. It is clearly the case that $C \subseteq A \cup B$, and then it follows directly that $\mu(C) \leq \mu(A) + \mu(B) = 0$. 