Homework set 10 — APPM5450, Spring 2013 — Solution to 11.16 convolution

We will prove that for \( \varphi \in \mathcal{S}(\mathbb{R}) \) and \( T = \mathcal{S}^*(\mathbb{R}) \), we have
\[
\mathcal{F}[\varphi * T] = \sqrt{2\pi} \hat{\varphi} \hat{T}.
\]

First note that this part of the problem is far harder than the other ones.

The trickiness is due to the indirect definition \([\varphi * T](x) = \langle T, R \tau_x \varphi \rangle\). Note also that since \( \varphi * T \) is not necessarily in \( L^1 \), we have to define \( \mathcal{F}[\varphi * T] \) in a distributional sense. So fix \( \psi \in \mathcal{S} \). Then
\[
\langle \mathcal{F}[\varphi * T], \psi \rangle = \langle T, R \tau_x \varphi \rangle \hat{\psi}(x) dx = \int_{\mathbb{R}} \varphi(x-y) \hat{\psi}(x) dx
\]
\[
\overset{(4)}{=} \langle T, (R \varphi) \hat{\psi} \rangle \overset{(5)}{=} \langle T, \mathcal{F} (\mathcal{F}^* ((R \varphi) \hat{\psi})) \rangle \overset{(6)}{=} \langle T, \mathcal{F} (\sqrt{2\pi} \hat{\varphi} \psi) \rangle
\]
\[
\overset{(7)}{=} \langle \hat{T}, \sqrt{2\pi} \hat{\varphi} \hat{\psi} \rangle \overset{(8)}{=} \langle \sqrt{2\pi} \hat{\varphi} \hat{T}, \psi \rangle
\]

Some comments on each step:

1. Simply the definition of the distributional Fourier transform.
2. Use that \( \varphi * T \) is a “plain” tempered function, and \( \hat{\varphi} \in \mathcal{S}(\mathbb{R}) \).
3. Here is des Pudels Kern. We need to move the integral inside the \( \mathcal{S}^* \times \mathcal{S} \) pairing. First observe that the integrand is smooth and rapidly decaying, so Riemann sums converge nicely. Consider a Riemann sum on the interval \([-n, n] \), with spacing \( 1/n \). Then
\[
\int_{\mathbb{R}} \langle T, R \tau_x \varphi \rangle \hat{\psi}(x) dx = \lim_{n \to \infty} \sum_{j=-n^2}^{n^2} \frac{1}{n} \langle T, R \tau_{j/n} \varphi \rangle \hat{\psi}(j/n)
\]
\[
= \lim_{n \to \infty} \int_{\mathbb{R}} \varphi(x-y) \hat{\psi}(y) dy
\]
In order to justify taking the limit inside the \( \mathcal{S}^* \times \mathcal{S} \) pairing, we invoke the continuity of \( T \), and then “only” need to prove that
\[
\sum_{j=-n^2}^{n^2} \frac{1}{n} \varphi(j/n - y) \hat{\psi}(j/n) \to \int_{\mathbb{R}} \varphi(x-y) \hat{\psi}(x) dx, \quad \text{as } n \to \infty,
\]
where the convergence is in \( \mathcal{S} \). We leave the details as an exercise. ☺
4. Observe that the function \( y \mapsto \int_{\mathbb{R}} \varphi(x-y) \hat{\psi}(x) dx \) is the convolution between \( R \varphi \) and \( \hat{\psi} \).
5. Use that \( \mathcal{F} \mathcal{F}^* = I \) on \( \mathcal{S} \).
6. Use that \( \mathcal{F}^*(f * g) = \sqrt{2\pi} (\mathcal{F}^* f) (\mathcal{F}^* g) \) for any \( f, g \in \mathcal{S} \), and that \( \mathcal{F}^* R = \mathcal{F} \).
7. Simply the definition of the distributional Fourier transform.
8. Definition of multiplication by a Schwartz function.