From the textbook: 11.5, 11.9, 11.15.

Problem 1: We say that a sequence \( (\varphi_n)_{n=1}^\infty \) is an approximate identity if

1. \( \varphi_n \in C(\mathbb{R}^d), \quad \forall \, n \),
2. \( \varphi_n(x) \geq 0, \quad \forall \, n, x \),
3. \( \int_{\mathbb{R}^d} \varphi_n(x) \, dx = 1, \quad \forall \, n \),
4. \( \forall \varepsilon > 0, \quad \int_{|x| \geq \varepsilon} \varphi_n(x) \, dx \to 0 \) as \( n \to \infty \).

(a) Do the conditions imply that \( \varphi_n \in S^* \)?

(b) Assuming that \( \varphi_n \in S^* \), prove that \( \varphi_n \to \delta \) in \( S^* \).

Problem 2: Compute the Fourier transforms of \( f(x) = \chi_{[-R,R]}(x) \) and \( f(x) = e^{-a|x|} \) by simply evaluating the formula

\[
\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx.
\]

The answers are given in examples 11.32 and 11.33 in the text book.

Problem 3 (optional): Let \( k \) be a positive integer. Prove that there exist numbers \( c_k \) and \( C_k \) such that \( 0 < c_k \leq C_k < \infty \), and

\[
c_k (1 + |x|^k) \leq (1 + |x|^{2k})^{k/2} \leq C_k (1 + |x|^k), \quad \forall \, x \in \mathbb{R}^d.
\]

Check to see if you can readily adapt your proof to also prove the existence of numbers \( b_k \) and \( B_k \) such that \( 0 < b_k \leq B_k < \infty \) such that

\[
b_k (1 + |x|)^k \leq (1 + |x|^{2k})^{k/2} \leq B_k (1 + |x|)^k, \quad \forall \, x \in \mathbb{R}^d.
\]

Note 1: The existence of inequalities such as (1) and (2) are routinely used (generally without even commenting on it) to replace the growth factor \( (1 + |x|^{2k})^{k/2} \) in the norms \( || \cdot ||_{a,k} \) by either \( (1 + |x|^k) \) or \( (1 + |x|)^k \), whenever convenient.

Note 2: If you have time, you may find it interesting to see what happens to the numbers \( b_k, B_k, c_k, C_k \) as \( k \) grows large. (This is easily done using Matlab.)