Exercise 7.13: Set $I = [0, 1]$ and consider the equation

$$i u_t = -u_{xx}, \quad x \in I, \quad t > 0,$$

for a complex valued function $u = u(x, t)$ with homogeneous boundary conditions,

$$u(0, t) = u(1, t) = 0,$$

and initial condition

$$u(x, 0) = f(x).$$

Set 

$$e_n(x) = \sqrt{2} \sin(n x).$$

Then $(e_n)_{n=1}^{\infty}$ forms an ON-basis for $L^2(I)$. We look for a solution

$$u(x, t) = \sum_{n=1}^{\infty} \alpha_n(t) e_n(x).$$

Inserting (4) into (1) and (3), we find that $\alpha_n$ must satisfy

$$i \alpha'_n = n^2 \alpha_n, \quad \alpha_n(0) = f_n,$$

where $f_n = (e_n, f)$. The solution is

$$\alpha_n(t) = f_n e^{-i n^2 t}.$$

Since $|\alpha_n(t)| = |f_n|$ for any $t$, it follows directly from Parseval that

$$||u(t)||^2_{L^2(I)} = \sum_{n=1}^{\infty} |\alpha_n(t)|^2 = \sum_{n=1}^{\infty} |f_n|^2 = ||f||^2,$$

and that (using that the cosines also form an ON-set)

$$||u_x(t)||^2_{L^2(I)} = \sum_{n=1}^{\infty} f_n e^{-i n^2 t} n \sqrt{2} \cos(n x)||^2_{L^2(I)} = \sum_{n=1}^{\infty} n |f_n|^2 = ||f_x||^2.$$

For a direct proof, set $v = \text{Re}(u)$ and $w = \text{Im}(u)$ so that $u = v + i w$. Then (1) takes the form

$$v_t = -w_{xx} \quad w_t = v_{xx}.$$

Now

$$\frac{d}{dt} \int_0^1 |u|^2 \, dx = \frac{d}{dt} \int_0^1 (v^2 + w^2) \, dx = 2 \int_0^1 (v_t v + w_t w) \, dx$$

$$= 2 \int_0^1 (-w_{xx} v + v_{xx} w) \, dx = 2 \int_0^1 (w_x v_x - v_x w_x) \, dx = 0.$$

The second to last step was partial integration where the boundary terms vanish due to (2). Analogously,

$$\frac{d}{dt} \int_0^1 |u_x|^2 \, dx = \frac{d}{dt} \int_0^1 (v_x^2 + w_x^2) \, dx = 2 \int_0^1 (v_{xt} v_x + w_{xt} w_x) \, dx$$

$$= 2 \int_0^1 (-v_t v_{xx} - w_t w_{xx}) \, dx = 2 \int_0^1 (-v_t w_t + w_t v_t) \, dx = 0.$$

In the second calculation we used differentiation, (2) takes the form

$$v_t(0, t) = v_t(1, t) = w_t(0, t) = w_t(1, t) = 0, \quad t > 0.$$
Exercise 8.3: Let $P$ and $Q$ be orthogonal projections. Set $M = \text{ran}(P)$ and $N = \text{ran}(Q)$. TFAE:

1. $M \subseteq N$
2. $QP = P$
3. $PQ = P$
4. $\|Px\| \leq \|Qx\| \quad \forall x$
5. $(x, Px) \leq (x, Qx) \quad \forall x$

Proof:

(a) $\Rightarrow$ (b): Assume $M \subseteq N$. Then for any $x$, $Px \in M \subseteq N$, so $QPx = Px$.

(b) $\Rightarrow$ (a): Assume $QP = P$. Pick $y \in M$. Then $y = Px$ for some $x$. Then $Qy = QPx = Px = y$ so $y \in N$.

(a) $\Leftrightarrow$ (c):

\[
M \subseteq N \quad \Leftrightarrow \quad N^\perp \subseteq M^\perp \\
\quad \Leftrightarrow \quad Py = 0 \quad \forall y \in N^\perp \\
\quad \Leftrightarrow \quad P(I - Q)x = 0 \quad \forall x \\
\quad \Leftrightarrow \quad P = PQ
\]

(c) $\Rightarrow$ (d): Assume $PQ = P$. Since $\|P\| \leq 1$ we have $\|Px\| = \|PQx\| \leq \|Qx\|$ for any $x$.

(d) $\Rightarrow$ (a): Assume that (a) is false. Then there is an $x \in M \setminus N$. Since $x \in M$ we have $x = Px$ and so

\[
\|Px\|^2 = \|x\|^2 = \|Qx + (I - Q)x\|^2 = \|Qx\|^2 + \|(I - Q)x\|^2.
\]

Now observe that $\|(I - Q)x\| > 0$ since $x \notin N$. Consequently,

\[
\|Qx\|^2 = \|Px\|^2 - \|(I - Q)x\|^2 < \|Px\|^2
\]

so (d) cannot hold true.

(d) $\Leftrightarrow$ (e): Simply observe that $(x, Px) = (x, P^2x) = (Px, Px) = \|Px\|^2$ and analogously $(x, Qx) = \|Qx\|^2$.

Note: You may want to draw a diagram over the implications to convince yourself that all equivalencies have been proven.
Exercise 8.4: First we prove that \( P_n \to I \) strongly. Fix any \( x \in H \). Since \( \bigcup_{n=1}^{\infty} \text{ran}(P_n) = H \), we know that \( x \in \text{ran}(P_N) \) for some specific \( N \). Then, since \( \text{ran}(P_n) \subseteq \text{ran}(P_{n+1}) \), we see that \( x \in \text{ran}(P_m) \) for any \( m \geq N \). Consequently, \( P_n x = x \) for any \( m \geq N \) so \( P_n x \to x \) (very rapidly!).

Next suppose that \( ||I - P_n|| \to 0 \). Then there is some \( N \) such that \( ||I - P_N|| \leq 1/2 \). Now observe that \( I - P_N \) is itself an orthogonal projection (onto \( \ker(P_N) \)) so it can only have norms 0 and 1. It follows that \( ||I - P_N|| = 0 \), which is to say that \( P_N = I \). Since \( H = \text{ran}(P_N) \subseteq \text{ran}(P_{N+1}) \subseteq \text{ran}(P_{N+2}) \subseteq \cdots \) we see that \( P_n = I \) for any \( n \geq N \).

Problem 1: Let \( T(t) \) denote the semigroup defined in Section 7.3 of the textbook. Prove that \( T(t) \to I \) strongly as \( t \downarrow 0 \). Prove that \( T(t) \) does not converge in norm.

Solution: We consider a slightly more general problem. Let \( (e_n)_{n=1}^{\infty} \) be an ON-basis for a Hilbert space \( H \), and consider for \( t \geq 0 \) the operator

\[
T(t)f = \sum_{n=1}^{\infty} f_n e^{-n^2 t} e_n.
\]

We will show that as \( t \downarrow 0 \), \( T(t) \to I \) strongly but not in norm.

To show \( T(t) \to I \) strongly, fix \( f \in H \). Fix \( \varepsilon > 0 \). Set \( f_n = (e_n, f) \) and pick \( N \) such that \( \sum_{n=N+1}^{\infty} |f_n|^2 < \varepsilon^2 \). Then by Parseval

\[
||T(t)f - f||^2 = \sum_{n=1}^{N} \left| f_n (e^{-n^2 t} - 1) \right|^2 + \sum_{n=N+1}^{\infty} \left| f_n (e^{-n^2 t} - 1) \right|^2
\]

\[
\leq \sum_{n=1}^{N} \left| f_n (e^{-n^2 t} - 1) \right|^2 + \sum_{n=N+1}^{\infty} 4 |f_n|^2 \leq \sum_{n=1}^{N} \left| f_n (e^{-n^2 t} - 1) \right|^2 + 4\varepsilon^2.
\]

Since only finitely many terms depend on \( t \), we can now easily take the limit as \( t \downarrow 0 \),

\[
\lim_{t \downarrow 0} ||T(t)f - f||^2 \leq 4 \varepsilon^2.
\]

Since \( \varepsilon \) was arbitrary, we see that \( \lim_{t \downarrow 0} ||T(t)f - f|| = 0 \).

To show that \( T(t) \) does not converge to \( I \) in norm, we simply observe that for any \( t > 0 \)

\[
||T(t) - I|| \geq \sup \left| (T(t) - I) e_n \right| = \sup_n |e^{-n^2 t} - 1| = 1.
\]
Problem 2: Suppose $P$ is a projection on a Hilbert space $H$. TFAE:

(1) $P$ is orthogonal, i.e. $\ker(P) = \overline{\text{ran}(P)}$.
(2) $P$ is self-adjoint, i.e. $\langle Px, y \rangle = \langle x, Py \rangle \ \forall x, y$.
(3) $\|P\| = 0$ or $1$.

Proof:

(a) $\Rightarrow$ (b): Assume $\ker(P) = \overline{\text{ran}(P)}$. Pick any $x, y \in H$. Then
\[
(Px, y) = \left( \sum_{x \in \text{ran}(P)} Px, y + \sum_{y \in \ker(P)} (I - P)y \right) = \sum_{x \in \text{ran}(P)} (Px, Py) = \sum_{y \in \ker(P)} (Px + (I - P)x, Py) = (x, Py).
\]

(b) $\Rightarrow$ (c): Assume that (b) holds. Then for any $x$,
\[
\|Px\|^2 = (Px, Px) = (P^2x, x) = (Px, x) \leq \|Px\| \|x\|,
\]
so $\|P\| \leq 1$. Obviously it is possible for $\|P\|$ to be zero. We need to prove that the only possible non-zero value of $\|P\|$ is one. To this end, note that if $P \neq 0$, then $\text{ran}(P) \neq \{0\}$. Now observe that if $x$ is a non-zero element in $\text{ran}(P)$, we have $Px = x$ so $\|P\| \geq 1$.

(c) $\Rightarrow$ (a): Assume that (a) does not hold. Then there exist $x \in \text{ran}(P)$ and $y \in \ker(P)$ such that $(x, y) \neq 0$. Set $\alpha = \frac{(x, y)}{|(x, y)|}$ and $z = \alpha y$. Then $z \in \ker(P)$ and $(x, z) = |(x, y)| \in \mathbb{R}_+$. Set $w = x - z t$.

Then $\|Pw\| = ||x||$, and
\[
|w|^2 = ||x||^2 - 2 t (x, z) + t^2 ||z||^2.
\]
For small $t$, we see that $|w| < ||x|| = ||Pw||$ so $\|P\| > 1$.

No solution is given for Problem 3 since the problem itself outlines precisely how to solve it — just fill in the details.