Problem 1: (12p) No motivation required for these problems.

(a) (3p) Let \( n^2 \in \mathbb{Z} \) and define \( f_n \in S^*(\mathbb{R}) \) via \( f_n(x) = \sin(nx) \). What is \( \hat{f}_n \)?

(b) (3p) State for which \( p \in [1, \infty] \), if any, the unit ball in \( L^p(\mathbb{R}) \) is weakly compact.

(c) (3p) Set \( H = L^2(\mathbb{R}) \) and define \( T \in \mathcal{B}(H) \) via \( [Tu](x) = u(-x) \). What is \( \sigma(T) \)?

(d) (3p) Let \( H \) be a Hilbert space. State the definition of a unitary operator on \( H \).

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Solution

(a) Observe that
- \( \sin(nx) = (1/2i) (e^{inx} - e^{-inx}) \),
- \( \mathcal{F} \delta = \beta \) (where \( \beta = 1/\sqrt{2\pi} \)),
- \( [\mathcal{F}(e^{inx} g)](t) = \hat{g}(t-n) \).

Combining, we find
\[
\hat{f}_n(t) = (1/2i) (\beta \delta(t-n) - \beta \delta(t+n)) = \frac{i}{2\sqrt{2\pi}} \delta(t+n) - \frac{i}{2\sqrt{2\pi}} \delta(t-n).
\]

(b) This is Banach-Alaoglu, which applies in reflexive spaces. Consequently, the unit ball is weakly compact when \( p \in (1, \infty) \).

(c) Observe that \( T \) is both unitary and self-adjoint. This means that the spectrum is contained in the intersection of the real line and the unit circle, which is to say \( \sigma(T) \subseteq \{-1, 1\} \). It is then easily verified that any even function is an eigenvector associated with \( \lambda = 1 \) and any odd function is an eigenvector associated with \( \lambda = -1 \). So \( \sigma(T) = \sigma_p(T) = \{-1, 1\} \).

(d) A unitary operator is a bijective operator that preserves the inner product.

Problem 2: (13p) Let \( H \) be a Hilbert space, and let \( A \) denote a bounded linear operator on \( H \).

(a) (3p) State the definition of the resolvent set \( \rho(A) \) of \( A \).

(b) (10p) Prove that the resolvent set \( \rho(A) \) is an open subset of \( \mathbb{C} \).

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Solution

(a) \( \rho(A) \) is the set of complex numbers \( \lambda \) such that \( A - \lambda I \) is one-to-one and onto.

(b) Fix \( \lambda \in \rho(A) \). Then \( A - \lambda I \) is continuously invertible by the open mapping theorem. Set \( \varepsilon = 1/\| (A - \lambda I)^{-1} \| \) and observe that \( \varepsilon > 0 \). For any \( \mu \in B_\varepsilon(\lambda) \), we find
\[
A - \mu I = A - \lambda I - (\mu - \lambda) I = (A - \lambda I) [I - (\mu - \lambda) (A - \lambda I)^{-1}].
\]

Now observe that
\[
\| (\mu - \lambda) (A - \lambda I)^{-1} \| \leq |\mu - \lambda| \| (A - \lambda I)^{-1} \| < \varepsilon \| (A - \lambda I)^{-1} \| = 1.
\]

Consequently, the Neumann series argument shows that the expression in brackets in (1) is invertible.
Problem 3: (16p) Define for \( \alpha, \beta \in (0, \infty) \) and for \( n = 1, 2, 3, \ldots \) functionals \( A_n, B_n \in S^*(\mathbb{R}) \) via

\[
A_n(\varphi) = \sum_{j=1}^{n} \alpha^j \varphi(j), \quad \text{and} \quad B_n(\varphi) = \sum_{j=1}^{n} j^\beta \varphi(j).
\]

(a) (8p) For which \( \alpha \in (0, \infty) \) does the sequence \( (A_n)_{n=1}^\infty \) converge in \( S^*(\mathbb{R}) \)?

(b) (8p) For which \( \beta \in (0, \infty) \) does the sequence \( (B_n)_{n=1}^\infty \) converge in \( S^*(\mathbb{R}) \)?

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**Solution**

**Answer:** For \( \alpha \in (0, 1) \) and for any \( \beta \in (0, \infty) \).

To prove that, e.g., \( (B_n) \) converges, we need to show that for every \( \varphi \in S \), the sequence \( (B_n(\varphi))_{n=1}^\infty \) converges to some number \( B(\varphi) \), where \( B \in S^* \).

To prove that \( (A_n) \) converges, we will show that there exists a \( \varphi \in S \), such that the sequence \( (B_n(\varphi))_{n=1}^\infty \) diverges.

- **Case 1:** \( \beta \in (0, \infty) \)

  Pick \( k \) such that \( k > \beta + 1 \). Then
  \[
  |B_n(\varphi)| \leq \sum_{j=1}^{\infty} j^\beta |\varphi(j)| \leq \sum_{j=1}^{\infty} j^\beta \frac{||\varphi||_{0,k}}{(1+j^2)^{k/2}} \sim ||\varphi||_{0,k} \sum_{j=1}^{\infty} j^{\beta-k} < \infty.
  \]

- **Case 2:** \( \alpha \in (0, 1] \)

  The proof is entirely analogous to Case 1 since the “weights” are bounded:
  \[
  |A_n(\varphi)| \leq \sum_{j=1}^{\infty} |\varphi(j)| \leq \sum_{j=1}^{\infty} \frac{||\varphi||_{0,2}}{1+j^2} \leq C ||\varphi||_{0,2}.
  \]

- **Case 3:** \( \alpha \in (1, \infty) \)

  Note that the weights grow exponentially in this case, which means that we cannot dominate the sum using a polynomial decay factor. We instead seek a Schwartz function \( \varphi \) such that \( \alpha^j \varphi(j) \to \infty \). To this end, pick \( \gamma \in (1, \alpha) \), and set
  \[
  \varphi(x) = \gamma^{-x^2/\sqrt{1+x^2}}.
  \]

  Then \( \varphi \in S(\mathbb{R}) \), but
  \[
  A_n(\varphi) = \sum_{j=1}^{n} \alpha^j \varphi(j) \sim \sum_{j=1}^{n} \alpha^j \gamma^{-j} = \sum_{j=1}^{n} (\alpha/\gamma)^j \to \infty.
  \]
Problem 4: (23p) Let $T$ denote the unit circle as usual, and define a function $f \in L^2(T)$ via $f(x) = x$, where $T$ is parameterized using $x \in [-\pi, \pi]$.

(a) (5p) What are the Fourier coefficients of $f$?

(b) (5p) For which $s \in [0, \infty)$ is it the case that $f \in H^s(T)$?

(c) (5p) Use your result in (a) to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

(d) (5p) Let $g$ denote the real-valued function obtained via periodic continuation of $f$ to a $2\pi$ periodic function on $\mathbb{R}$. Prove that $g \in S^s(\mathbb{R})$.

(e) (3p) What is the Fourier transform of the function $g \in S^s(\mathbb{R})$ defined in (d)?

No motivation required for this part. (Hint: Problem 1(a) may be useful.)

Solution

(a) Set $\beta = 1/\sqrt{2\pi}$. Then $\alpha_n = \beta \int_{-\pi}^{\pi} e^{-inx} x \, dx = \beta i \int_{-\pi}^{\pi} \sin(nx) x \, dx = \cdots = \frac{2\beta i \pi (-1)^n}{n}$.

(b) We find $\|f\|_{L^2}^2 = \sum (1 + |n|^2) \alpha_n^2 = \sum (1 + |n|^2) \frac{4\beta^2 \pi^2}{n^2} \sim \sum n^{2s} n^{-2}$. The sum is finite iff $2s - 2 < -1$, which is to say $s < 1/2$.

(c) Parseval’s theorem states that $\|f\|_{L^2}^2 = \sum |\alpha_n|^2$. Now

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 = 2 \sum_{n=1}^{\infty} \frac{4\beta^2 \pi^2}{n^2} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and

$$\|f\|_{L^2}^2 = \int_{-\pi}^{\pi} x^2 \, dx = 2 \int_{0}^{\pi} x^2 \, dx = \frac{2}{3} \pi^3.$$

(d) For a given $\varphi \in S$, we can bound $T_f$ as follows:

$$|T_f(\varphi)| = \left| \int_{-\infty}^{\infty} f(x) \varphi(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)| \|\varphi\|_{0.2} \, dx \leq \int_{-\infty}^{\infty} \pi |\varphi|_{0.2} \, dx = \pi^2 \|\varphi\|_{0.2}.$$

(e) We have $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n \beta e^{inx}$. Since $[Fe^{inx}](t) = \beta \delta(t - n)$, we get

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} \alpha_n \beta^2 \delta(t - n) = \sum_{n=-\infty}^{\infty} \frac{2\beta i \pi (-1)^n}{n} \beta^2 \delta(t - n) = \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n \sqrt{2\pi}} \delta(t - n).$$

We treated the sum in a cavalier manner, but we only needed the answer!

Note: The Fourier sum simplifies as $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$. The first 20 terms look like:
**Problem 5:** (18p) Set \( I = (0, 1) \) and let \((f_n)_{n=1}^{\infty}\) be a sequence of Lebesgue integrable real valued functions on the interval \( I = (0, 1) \) such that for every \( x \in I \),
\[
\lim_{n \to \infty} f_n(x) = x.
\]
Consider for \( n = 1, 2, 3, \ldots \) the three sequences
\[
a_n = \int_0^1 f_n(x) \, dx
\]
\[
b_n = \int_0^1 \frac{f_n(x)}{1 + (f_n(x))^2} \, dx
\]
\[
c_n = \int_0^1 \left| \sum_{j=1}^n f_j(x) \right| \, dx.
\]
Which of the sequences must necessarily converge as \( n \to \infty \)? Is it for any of the convergent sequences possible to say what the limit is? Motivate your answers.

**Solution**

The sequence \( a_n \): This may or may not converge.  
If say \( f_n(x) = x \) for all \( x \), then \( a_n \to 1/2 \).  
If on the other hand \( f_n = n^2 \chi(0,1/n) + x \chi(1/n,1) \), then \( a_n \to \infty \).

The sequence \( b_n \): The absolute value of the integrand is bounded by \( g(x) = 1 \). Since \( 0^1 g \, dx = 1 \) is finite, Lebesgue dominated convergence applies and we find that
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \int_0^1 \frac{f_n(x)}{1 + (f_n(x))^2} \, dx = \int_0^1 \lim_{n \to \infty} \frac{f_n(x)}{1 + (f_n(x))^2} \, dx = \int_0^1 \frac{x}{1 + x^2} \, dx
\]
\[
= \left[ \frac{1}{2} \log(1 + x^2) \right]_0^1 = \frac{1}{2} (\log(2) - \log(1)) = \frac{\log(2)}{2}.
\]

The sequence \( c_n \): Since the integrand is non-negative, Fatou’s lemma applies:
\[
\liminf_{n \to \infty} c_n = \liminf_{n \to \infty} \int_0^1 \left| \sum_{j=1}^n f_j(x) \right| \, dx \geq \int_0^1 \liminf_{n \to \infty} \left| \sum_{j=1}^n f_j(x) \right| \, dx.
\]
For any \( x \), we have \( \sum_{j=1}^n f_j(x) \to \infty \), so \( \liminf_{n \to \infty} \left| \sum_{j=1}^n f_j(x) \right| = \infty \), and consequently \( c_n \to \infty \).
Problem 6: (18p) Let \((f_n)_{n=1}^\infty\) be a sequence of functions in \(L^2(\mathbb{R})\) that converges pointwise to a function \(f\). In other words,
\[
\lim_{n \to \infty} f_n(x) = f(x), \quad \text{for all } x \in \mathbb{R}.
\]
Suppose further that all \(f_n\) satisfy
\[
|f_n(x)| \leq 2|f(x)|, \quad \text{for all } x \in \mathbb{R}.
\]
For each of the three sets of conditions on \(f\) given below, specify for which \(r \in [1, \infty)\) it is necessarily the case that
\[
\lim_{n \to \infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.
\]
(a) \((6p)\) \(f \in L^2(\mathbb{R})\), and for \(|x| \geq 2\), it is the case that \(f(x) = 0\).
(b) \((6p)\) \(f \in L^2(\mathbb{R})\) and \(|f(x)| \leq 2\) for all \(x \in \mathbb{R}\).
(c) \((6p)\) \(f \in L^2(\mathbb{R})\) and \(f \in L^3(\mathbb{R})\).

\underline{Solution}

**Answers:** (a) \(r \in [1, 2]\). (b) \(r \in [2, \infty)\). (c) \(r \in [2, 3]\).

We need to prove the claim when it is true, and provide counter-examples when it is not. The basic question we need to resolve is when
\[
(2) \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} |f(x) - f_n(x)|^r \, dx = 0.
\]
The integrand in (2) converges to zero pointwise, and we want to bring the LDCT to bear. To this end, we construct a dominator \(h\) via
\[
|f(x) - f_n(x)|^r \leq (|f(x)| + |f_n(x)|)^r \leq (|f(x)| + 2|f(x)|)^r = 3^r |f(x)|^r =: h(x).
\]
We will analyze each of the three assumptions to see when \(\int h < \infty\).

(a) Suppose \(r \in [1, 2]\). Then \(h(x) = 3^r |f(x)|^r \leq 3^r \max(1, |f(x)|^2)\). Since \(f \in L^2\), and since in this case, \(h\) has compact support, we find \(\int h < \infty\).

Suppose \(r > 2\). When \(|f(x)| > 1\), we have \(|f(x)|^r > |f(x)|^2\), so \(h\) does not necessarily have finite integral and the LDCT does not apply. We look for a counter-example. Pick a real number \(\alpha\) such that \(-\frac{1}{2} < \alpha < -\frac{1}{4}\), and set \(f(x) = x^\alpha \chi_{(0,1)}\). Then \(f \in L^2\). Set \(f_n = (1 - 1/n) f\). Then \(f_n \to f\) pointwise, but \(||f - f_n||_r^r = ||(1/n)f||_r^r = \int_0^1 n^{-r} x^{\alpha r} \, dx = \infty\).

(b) Suppose \(r \in [2, \infty)\). Then \(h(x) = 3^r |f(x)|^r \leq 6^r |f(x)/2|^r \leq 6^r |f(x)/2|^2\), since \(|f(x)/2| \leq 1\) and \(r \geq 2\). We find \(\int h \leq 6^r (1/4)||f||_2^2 < \infty\), so LDCT applies.

Suppose \(r \in [1, 2]\). In this case, the LDCT does not apply, and we look for a counter-example. Pick a real number \(\alpha\) such that \(-\frac{1}{2} < \alpha < -\frac{1}{4}\), and set \(f(x) = x^\alpha \chi_{[1, \infty)}\). Then \(f \in L^2\). Set \(f_n = (1 - 1/n) f\). Then \(f_n \to f\) pointwise, but \(||f - f_n||_r^r = ||(1/n)f||_r^r = \int_1^\infty n^{-r} x^{\alpha r} \, dx = \infty\).

(c) Suppose \(r \in [2, 3]\). Then by interpolation (see Homework 14 – Problem 12.15), \(f \in L^r\). It follows that \(\int h < \infty\), and so the LDCT applies.

Suppose \(r < 2\). In this case, construct a counter-example as in part (b) of a function \(f\) that does not decay fast enough to belong to \(L^r\).

Suppose \(r > 3\). In this case, construct a counter-example as in part (a) of a function \(f\) that has a sufficiently strong singularity that it does not belong to \(L^r\).