Applied Analysis (APPM 5450): Midterm 2 — Solutions

The problems are worth 20 points each. Briefly motivate all answers except those to Problem 1.

**Problem 1:** No motivation required for these questions.

(a) Give an example of a bounded linear operator on a Hilbert space that is positive, but not coercive.

(b) Let $H$ be an infinite dimensional Hilbert space. Which of the following sets can be the spectrum of a compact self-adjoint operator?

1. $A_1 = \{1/n\}_{n=1}^\infty = \{1, 1/2, 1/3, 1/4, \ldots \}$
2. $A_2 = \{1\} \cup \{1 - 1/n\}_{n=1}^\infty = \{1, 0, 1/2, 2/3, 3/4, 4/5, \ldots \}.$
3. $A_3 = \{0, 1\} \cup \{e^{i/n}\}_{n=1}^\infty = \{0, 1, e^i, e^{i/2}, e^{i/3}, e^{i/4}, \ldots \}.$
4. $A_4 = \{1, 2, 3\}.$
5. $A_5 = \{-1, 0\}.$

(c) Define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = e^{-x^2}.$ What is $\langle \delta'' , \varphi \rangle$?

(d) Define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = e^{-x^2}.$ What is $\delta'' * \varphi$?

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**Solution:**

(a) There are obviously many possible examples. A couple of simple ones:

- $H = L^2(I)$ where $I = [0, 1]$ and $[Au](x) = xu(x)$.
- $H = \ell^2(\mathbb{N})$ and $A(x_1, x_2, x_3, x_4, \ldots) = (\frac{1}{1} x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \frac{1}{4} x_4, \ldots)$

(b) Only $A_5$. (Grading guide: -2p for each mistake.)

  (a) $A_1$ does not include zero (and is also not closed).
  (b) $A_2$ has an accumulation point at 1.
  (c) $A_3$ is not a subset of the real line.
  (d) $A_4$ does not include zero.
  (e) Let $u$ be a non-zero vector in $H$ and set $Ax = -\frac{1}{||u||^2} (u, x) u$.

      Then $A$ is self-adjoint and compact, and $\sigma(A) = A_5$.

(c) $-2$

(d) The function $x \mapsto \varphi''(x) = (4x^2 - 2)e^{-x^2}$.
**Problem 2:** Set $H = \ell^2(\mathbb{Z})$ and let $A \in \mathcal{B}(H)$ denote the rightshift operator (i.e. if $u \in H$ and $v = A u$, then $v_n = u_{n-1}$).

(a) Let $\lambda$ be a complex number such that $|\lambda| = 1$. Prove that you can construct $u^{(n)} \in H$ such that $\|u^{(n)}\| = 1$ and $\lim_{n \to \infty} \|A u^{(n)} - \lambda u^{(n)}\| = 0$.

(b) Determine the spectrum of $A$.

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**Solution:**

(a) Suppose $|\lambda| = 1$ and set $R_\lambda = A - \lambda I$. First verify that $R_\lambda$ is injective by noting that if $R_\lambda u = 0$, then $u_n = \lambda^{-n} u_0$ which implies that $|u_n| = |u_0|$ for all $n$. The only solution is therefore $u = 0$. Next observe that the range of $R_\lambda$ is dense since

$$\text{ran}(A - \lambda I) = (\ker(A^* - \lambda I))^\perp = \{0\}^\perp = H.$$  

(The proof that $A - \lambda I$ is injective immediately carries over to a proof that $A^* - \lambda I$ is injective since $A^*$ is simply left-shift.) Finally observe that $A - \lambda I$ is not onto since, e.g., the zero'th canonical basis vector $e^{(0)}$ does not belong to the range.\(^1\) The closed range theorem now implies that $R_\lambda$ cannot be coercive since its range is not closed.

(b) Set

$$D = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$  

We proved in part (a) that $D \subseteq \sigma_c(A)$. Observe next that $A$ is a unitary operator. It follows\(^2\) that $\sigma(A) \subseteq D$ and consequently

$$\sigma(A) = \sigma_c(A) = D \quad \sigma_p(A) = \sigma_t(A) = \emptyset.$$  

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*Alternative explicit proof:* Let $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ denote the standard Fourier transform. We will exploit that $\mathcal{F}$ is unitary, and consequently the operator $T = F^* A F$ has the same spectral properties as $A$. A simple calculation shows that

$$[T U](x) = e^{ix} U(x).$$  

Given a $\lambda$ such that $|\lambda| = 1$, pick $\theta$ such that $\lambda = e^{i\theta}$. Then set

$$U^{(n)}(x) = \left\{ \begin{array}{ll} \sqrt{\frac{\pi}{2}} & \text{when } |x - \theta| \leq 1/n \\ 0 & \text{when } |x - \theta| > 1/n. \end{array} \right.$$  

It follows that $\|U^{(n)}\| = 1$ and $\lim_{n \to \infty} \|T U^{(n)} - \lambda U^{(n)}\| = 0$. Now set $u^{(n)} = F U^{(n)}$.

\(^1\)To prove this, suppose $A u - \lambda u = e^{(0)}$. Then for non-zero $n$, we have $u_{n-1} = \lambda u_n$ so $u_n = \lambda^{-n} u_{-1}$ for negative $n$ and $u_n = \lambda^{-n} u_0$ for positive $n$. The only way for $u$ to be in $H$ is for $u$ to be the zero vector which is impossible.

\(^2\)The explicit proof is simple: For $|\lambda| > 1$ observe that $A - \lambda I = -\lambda (I - \lambda^{-1} A)$ and now the inverse can explicitly be constructed via a Neumann series since $\|\lambda^{-1} A\| = |\lambda|^{-1} < 1$. Analogously, if $|\lambda| < 1$, then $A - \lambda I = A (I - \lambda A^*)$ which is invertible since $A$ is invertible and since $\|\lambda A^*\| = |\lambda| < 1$. 


Problem 3: Define $T \in S^*(\mathbb{R})$ via
\[
\langle T, \varphi \rangle = \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x) \, dx.
\]
Construct a continuous function $f$ of at most polynomial growth such that $T = \partial^p f$ for some finite integer $p$.

Solution: First we integrate the function $1/x$ in a classical sense to find a candidate for a distributional primitive function.
\[
\int \int \frac{1}{x} = \int (\log |x| + A) = x \log |x| - x + Ax + B
\]
Set $A = 1$ and $B = 0$ to obtain the candidate
\[
f(x) = x \log |x|.
\]
The function $f$ is continuous and of polynomial growth. It remains to prove that $f'' = T$ in a distributional sense.
\[
\langle f'', \varphi \rangle = \langle f, \varphi'' \rangle
\]
\[

\begin{align*}
\stackrel{(1)}{=}& \lim_{\varepsilon \searrow 0} \left( \int_{-\infty}^{-\varepsilon} f' \varphi'' + \int_{\varepsilon}^{\infty} f \varphi'' \right) \\
\stackrel{(2)}{=}& \lim_{\varepsilon \searrow 0} \left( [f \varphi']_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} f' \varphi' + [f \varphi']_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty} f' \varphi' \right) \\
\stackrel{(3)}{=}& \lim_{\varepsilon \searrow 0} \left( - \int_{-\infty}^{-\varepsilon} f' \varphi' - \int_{\varepsilon}^{\infty} f' \varphi' \right) \\
\stackrel{(4)}{=}& \lim_{\varepsilon \searrow 0} \left( - \int_{-\infty}^{-\varepsilon} f' \varphi' - \int_{-\infty}^{-\varepsilon} f'' \varphi - [f \varphi']_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty} f'' \varphi \right) \\
\stackrel{(5)}{=}& \lim_{\varepsilon \searrow 0} \left( - \log(\varepsilon) \varphi(-\varepsilon) + \log(\varepsilon) \varphi(\varepsilon) \right) + \langle T, \varphi \rangle.
\end{align*}
\]
Relation (1) holds since the integrand is continuous.
Relation (2) is plain partial integration.
Relation (3) holds since $f \varphi'$ is a continuous function.
Relation (4) is plain partial integration.
Relation (5) holds since $f''(x) = 1/x$ in the domains of integration.
(Note that all limits at $\pm \infty$ vanish since $f \varphi'$ and $f' \varphi'$ both tend to zero since $\varphi \in S$ and $f$ and $f'$ have at most polynomial growth.)

Finally we observe that
\[
\lim_{\varepsilon \searrow 0} \left( - \log(\varepsilon) \varphi(-\varepsilon) + \log(\varepsilon) \varphi(\varepsilon) \right) = \lim_{\varepsilon \searrow 0} \log(\varepsilon) \left( \varphi(\varepsilon) - \varphi(-\varepsilon) \right) = 0
\]
since
\[
\left| \varphi(\varepsilon) - \varphi(-\varepsilon) \right| \leq 2 \varepsilon \|\varphi'\|_u
\]
and
\[
\lim_{\varepsilon \searrow 0} \varepsilon \log(\varepsilon) = 0.
\]
**Problem 4:** Fix $\psi \in S(\mathbb{R})$. Define the map 

$$B : S(\mathbb{R}) \to \mathbb{C} : \varphi \mapsto \int_{\mathbb{R}} \psi(x) \varphi'(x) \, dx.$$ 

Prove that $B$ is continuous. What order is $B$?

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**Solution:**

First observe that via a partial integration we can rewrite 

$$B(\varphi) = -\int_{-\infty}^{\infty} \psi'(x) \varphi(x) \, dx.$$ 

Then 

$$|B(\varphi)| = \left| -\int_{-\infty}^{\infty} \psi'(x) \varphi(x) \, dx \right| \leq \int_{-\infty}^{\infty} |\psi'(x)||\varphi(x)| \, dx \leq ||\psi'||_{L^1} ||\varphi||_{0,0}.$$ 

Observe that $||\psi'||_{L^1}$ is finite\(^3\) since $\psi \in S$ so $B$ is continuous and has order zero.

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\(^3\)To be precise $||\psi'||_{L^1} = \int |\psi'| \leq \int (1 + x^2) ||\psi||_{0,2} = \pi ||\psi||_{0,2} < \infty.$
Problem 5: Set \( H = L^2(\mathbb{T}) \) and define \( W \in B(H) \) via
\[
[W u](x) = \int_{-\pi}^{\pi} \sin(x - y) u(y) \, dy.
\]
Compute the spectrum of \( W \) and identify its different components (i.e. determine \( \sigma_p(W), \sigma_c(W), \) and \( \sigma_r(W) \)). Is \( W \) compact? Self-adjoint?

Solution: We define the canonical basis for \( H \) via
\[
e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z},
\]
and the corresponding canonical projections \( P_n \) via
\[
[P_n u](x) = e_n(x) \langle e_n, u \rangle = e^{inx} \int_{-\pi}^{\pi} e^{-iny} u(y) \, dy.
\]
Then observe that \( W \) can be written
\[
[W u](x) = \int_{-\pi}^{\pi} \frac{e^{i(x-y)} - e^{-i(x-y)}}{2i} u(y) \, dy
\]
\[
= \frac{e^{ix}}{2i} \int_{-\pi}^{\pi} e^{-iy} u(y) \, dy - \frac{e^{-ix}}{2i} \int_{-\pi}^{\pi} e^{iy} u(y) \, dy = -i\pi [P_1 u](x) + i\pi [P_{-1} u](x).
\]
It follows that
\[
\sigma(W) = \sigma_p(W) = \{0, i\pi, -i\pi\},
\]
and consequently \( \sigma_c(W) = \sigma_r(W) = \emptyset \).

Alternative solution: Recalling the trig identity
\[
\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)
\]
we write
\[
[W u](x) = \sin(x) \int_{-\pi}^{\pi} \cos(y) u(y) \, dy - \cos(x) \int_{-\pi}^{\pi} \sin(y) u(y) \, dy.
\]
Defining two orthonormal unit vectors \( v_1 \) and \( v_2 \) via
\[
v_1(x) = \frac{1}{\sqrt{\pi}} \sin(x), \quad v_2(x) = \frac{1}{\sqrt{\pi}} \cos(x),
\]
we can therefore write \( W \) as
\[
W u = \pi v_1 \langle v_2, u \rangle - \pi v_2 \langle v_1, u \rangle.
\]
Now set \( G = \text{span}\{v_1, v_2\} \) and observe that both \( G \) and \( G^\perp \) are invariant subspaces of \( W \). The restriction of \( W \) to \( G \) has the matrix
\[
W = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}
\]
and \( W \) has the eigenvalues \( \pm i\pi \). The restriction of \( W \) to \( G^\perp \) is zero. Therefore
\[
\sigma(W) = \sigma_p(W) = \{0, i\pi, -i\pi\}.
\]