Problem 1: (21p) All operators in this problem are bounded linear operators on a Hilbert space. Which statements are necessarily true? No motivation required.

(a) Every bounded sequence in a Hilbert space has a weakly convergent subsequence.

(b) If $A$ and $B$ are self-adjoint operators, then $A + B$ is self-adjoint.

(c) If $A$ and $B$ are self-adjoint operators, then $AB$ is self-adjoint.

(d) If $A$ and $B$ are unitary operators, then $A + B$ is unitary.

(e) If $A$ and $B$ are unitary operators, then $AB$ is unitary.

(f) If $A$ is skew-symmetric, then the operator $B = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is unitary.

(g) If $A$ is an isometric operator, then ran$(A) = (\ker(A^*))^\perp$.

Problem 2: (29p) Let $H_1$ denote the Hilbert space obtained by taking the completion of the set $P$ of trigonometric polynomials with respect to the norm induced by the inner product

$$\langle u, v \rangle_1 = \int_{-\pi}^{\pi} u(x) v(x) \, dx$$

and let $H_2$ denote the Hilbert space induced by the inner product

$$\langle u, v \rangle_2 = \int_{-\pi}^{\pi} u(x) v(x) (1 - \cos(x)) \, dx.$$ 

(a) Do the spaces $H_1$ and $H_2$ contain the same [equivalence classes of] functions?

(b) Does there exist a unitary map between $H_1$ and $H_2$?

(c) For which real numbers $\alpha$ is it the case that the sequence $(\varphi_n)_{n=1}^{\infty}$ where $\varphi_n = n^\alpha \chi_{(-1/n,1/n)}$ converges in norm in $H_1$? Is the answer different if you consider weak convergence?

(d) Repeat question (c), but now do the exercise in $H_2$.

(e) Set $\rho_n(x) = \sin(nx)$. Does the sequence $(\rho_n)_{n=1}^{\infty}$ converge in either $H_1$ or $H_2$? Weakly? In norm?

Problem 3: (20p) Set $f(t) = |t|$ for $-\pi \leq t < \pi$ and extend $f$ to be a $2\pi$-periodic function. Is it the case that $f \in H^k(\mathbb{T})$ for any $k \geq 0$?

Hint: The Sobolev embedding theorem should very quickly provide at least a partial answer.

Problem 4: (30p) Suppose that $P$ is a projection on a Hilbert space $H$. Prove that the following are equivalent:

(i) $P$ is orthogonal, i.e. $\ker(P) = \text{ran}(P)^\perp$.

(ii) $P$ is self-adjoint, i.e. $\langle Px, y \rangle = \langle x, Py \rangle$ $\forall x, y$.

(iii) $\|P\| = 0$ or 1.