11.4) If $\phi \in S(R)$, prove that $\phi \delta' = \phi(0)\delta' - \phi'(0)\delta$.

$$
\langle \phi \delta', \psi \rangle = \langle \delta', \phi \psi \rangle = -\langle \delta, (\phi \psi)' \rangle = -\langle \delta, \phi' \psi + \phi \psi' \rangle = -\langle \delta, \phi' \psi \rangle - \langle \delta, \phi \psi' \rangle = -\phi'(0)\psi(0) - \phi(0)\psi'(0) =
$$

$$
= -\phi'(0)\delta, \psi \rangle + \langle \phi(0)\delta', \psi \rangle = \langle -\phi'(0)\delta + \phi(0)\delta', \psi \rangle
$$

Note that in the equality across the line break the following was used: $-\psi'(0) = -\langle \delta, \psi' \rangle = \langle \delta', \psi \rangle$
11.9) Let \( \psi \in S \) and define the convolution operator \( Kf(x) = \int \psi(x-y)f(y)dy \) for all \( f \in S \).
Prove that \( K : S \to S \) is a continuous linear operator for the topology of \( S \).

Pick \( \phi \in S \). Then

\[
\|K\phi\|_{\alpha,k} = \sup_x \left| (1 + |x|^2)^{\|/2} \partial_\alpha \int \psi(x-y)\phi(y)dy \right| = \sup_x \left| (1 + |x|^2)^{\|/2} \int \psi^{(\alpha)}(x-y)\phi(y)dy \right| = (1)
\]

Next we introduce the substitution \( z = y - \frac{x}{2} \)

\[
(1) = \sup_x \left( (1 + |x|^2)^{\|/2} \int \psi^{(\alpha)} \left( \frac{x}{2} - z \right) \phi \left( \frac{x}{2} + z \right) dy \right) \leq \sup_x \left( (1 + |x|^2)^{\|/2} \int \psi_{\alpha,2N}^N \phi_{0,2N}^N dz \right) = (2)
\]

Note that in the step above where the \( \leq \) is we pick \( N \) s.t. \( N \geq k, N \geq d + 1 \).

We can bound the denominator (not including the exponent) as follows

\[
\left( 1 + \left| \frac{x}{2} - z \right|^2 \right) \left( 1 + \left| \frac{x}{2} + z \right|^2 \right) = 1 + \frac{1}{16} |x|^4 + |z|^4 + \frac{1}{2} |x|^2 |2|z|^2 + \frac{1}{2} |x|^2 |z|^2 - |x \cdot z|^2 \geq \\
\geq 1 + \frac{1}{2} |x|^2 + 2 |z|^2 + \frac{1}{16} |x|^4 + |z|^4 - \frac{1}{2} |x|^2 |z|^2 \geq 1 + \frac{1}{2} |x|^2 + 2 |z|^2 \\
\geq \left( \frac{1}{4} |x|^2 - |z|^2 \right) \geq 0
\]

Continuing from above we have

\[
(2) \leq \sup_x \left( (1 + |x|^2)^{\|/2} \int \psi_{\alpha,2N}^N \phi_{0,2N}^N dz \right) \leq \\
\leq \sup_x \left( (1 + |x|^2)^{\|/2} \int \psi_{\alpha,2N}^N \phi_{0,2N}^N dz \right) = C \|\psi\|_{\alpha,2N} \|\phi\|_{0,2N}
\]

Combining everything we have \( \|K\phi\|_{\alpha,k} \leq C \|\psi\|_{\alpha,2N} \|\phi\|_{0,2N} \)

Thus \( K : S \to S \) is a continuous linear operator for the topology of \( S \).
11.10) For every $h \in \mathbb{R}^n$ define a linear transform $\tau_h : S \to S$ by $\tau_h(f)(x) = f(x-h)$.

a) Prove that for all $h \in \mathbb{R}^n$, $\tau_h$ is continuous in the topology of $S$.
Assume $\phi_n \to \phi$ in $S$.
$$\left\| \tau_h(\phi_n) - \tau_h(\phi) \right\|_{\alpha,h} = \sup_x \left( 1 + |x|^2 \right)^{\frac{\alpha}{2}} \left( \partial^\alpha \phi_n(x - h) - \partial^\alpha \phi(x - h) \right) = \sup_x \left( 1 + |x + h|^2 \right)^{\frac{\alpha}{2}} \left( \partial^\alpha \phi_n(x) - \partial^\alpha \phi(x) \right) = (*)$$
Where the final equality above substitutes $x + h$ for $x$.
We can bound this as follows
$$1 + |x + h|^2 \leq 1 + (|x| + |h|)^2 \leq 1 + |x|^2 + |h|^2 + 2 |x||h| \leq 1 + 2|x|^2 + 2|h|^2 \leq 2\left(1 + |x|^2\right)(1 + |h|^2)$$
Using this bound and continuing from (*) we have
$$\left\| \tau_h(\phi_n) - \tau_h(\phi) \right\|_{\alpha,h} = \cdots = (*) \leq \sup_x \left( 2\left(1 + |x|^2\right)(1 + |h|^2) \right)^{\frac{\alpha}{2}} \left( \partial^\alpha \phi_n(x) - \partial^\alpha \phi(x) \right) = \left(2\left(1 + |h|^2\right)\right)^{\frac{\alpha}{2}} \left\| \phi_n - \phi \right\|_{\alpha,h} \to 0$$
The convergence is in the last step comes from the assumption that $\phi_n \to \phi$ in $S$.

b) Prove that for all $f \in S$, the map $h \mapsto \tau_h f$ is continuous from $\mathbb{R}^n$ to $S$.
Assume $h \to 0$ in $\mathbb{R}^n$. Then
$$\left\| \tau_h \phi - \phi \right\|_{\alpha,h} = \left\| \left(1 + |x|^2\right)^{\frac{\alpha}{2}} \partial^\alpha \phi(x-h) - \partial^\alpha \phi(x) \right\| \leq \left\| \left(1 + |x|^2\right)^{\frac{\alpha}{2}} \nabla^\alpha \phi(x) \right\| \leq |h| C \sum_{|\beta| = |\alpha| + 1} \left\| \phi \right\|_{\beta,k} \xrightarrow{h \to 0} 0$$
Note that above $x_n$ is some point on the line from $x - h$ to $x$ and the first inequality uses the mean value theorem for integrals.
Problem 1) We say that a sequence \((\phi_n)_{n=1}^\infty\) is an approximate identity if

1) \(\phi_n \in C(\mathbb{R}^d), \forall n\)
2) \(\phi_n(x) \geq 0, \forall n, x\)
3) \(\int_{\mathbb{R}^d} \phi_n(x) dx = 1, \forall n\)
4) \(\forall \varepsilon > 0, \int_{|x| \leq \varepsilon} \phi_n(x) dx \rightarrow_{n \to \infty} 0\)

a) Do the conditions imply that \(\phi_n \in S^*\)?
Yes. Conditions (1)-(3) above imply that \(\phi_n \in L^1\), and this immediately implies \(\phi_n \in S^*\).

b) Assuming that \(\phi_n \in S^*\), prove that \(\phi_n \to \delta\) in \(S^*\). Fix \(\varepsilon > 0\).
\[
\|\phi_n, \phi\| - \phi(0) = \left| \int_{\mathbb{R}^d} \phi_n(x) \phi(x) dx - \phi(0) \right| = \left| \int_{|x| \leq \varepsilon} \phi_n(x) (\phi(x) - \phi(0)) dx \right|
\]
\[
= \left| \int_{|x| \leq \varepsilon} \phi_n(x) (\phi(x) - \phi(0)) dx + \int_{|x| > \varepsilon} \phi_n(x) (\phi(x) - \phi(0)) dx \right|
\]
\[
\leq \int_{|x| \leq \varepsilon} \phi_n(x) (|\phi(x) - \phi(0)| dx + \int_{|x| > \varepsilon} \phi_n(x) (|\phi(x) - \phi(0)| dx
\]
\[
\leq \|\phi_n\|_\infty \varepsilon + \|\phi\|_0
\]
Note that the equality denoted by (a) above uses condition (3).
We now have
\[
\|\phi_n, \phi\| - \phi(0) \leq \varepsilon \|\phi\|_0 + 2 \|\phi\|_0 \left| \int_{|x| \leq \varepsilon} \phi_n(x) dx \right|
\]
This implies \(\lim_{n \to \infty} \|\phi_n, \phi\| - \phi(0) \leq \varepsilon \|\phi\|_0\)
Since \(\varepsilon\) was arbitrary we get \(\|\phi_n, \phi\| - \phi(0) \to 0\), or simply \(\phi_n \to \delta\) in \(S^*\).
Problem 3) Let \( k \) be a positive integer. Prove that there exist \( c_k, C_k \) s.t. \( 0 < c_k \leq C_k < \infty \), and

\[
(1) \quad c_k \left(1 + |x|^k\right) \leq \left(1 + |x|^2\right)^{k/2} \leq C_k \left(1 + |x|^k\right), \quad \forall x \in \mathbb{R}^d
\]

Prove that there exist \( b_k, B_k \) s.t. \( 0 < b_k \leq B_k < \infty \), and

\[
(2) \quad b_k \left(1 + |x|^k\right) \leq \left(1 + |x|^2\right)^{k/2} \leq B_k \left(1 + |x|^k\right), \quad \forall x \in \mathbb{R}^d
\]

To prove (1) we need to prove the following

\[
(a) \quad \sup_{x \in \mathbb{R}^d} \left(\frac{1 + |x|^2}{1 + |x|^k}\right)^{k/2} < \infty \quad \Leftrightarrow \quad \sup_{0 < r < \infty} \left(\frac{1 + r^2}{1 + r^k}\right)^{k/2} < \infty
\]

\[
(b) \quad \inf_{x \in \mathbb{R}^d} \left(\frac{1 + |x|^2}{1 + |x|^k}\right)^{k/2} > 0 \quad \Leftrightarrow \quad \inf_{0 < r < \infty} \left(\frac{1 + r^2}{1 + r^k}\right)^{k/2} > 0
\]

Set \( f(r) = \left(\frac{1 + r^2}{1 + r^k}\right)^{k/2} \). Then \( f(0) = 1 \) and \( f(\infty) = 1 \).

Since \( f \) is continuous and \( f(0) = 1 \) and \( f(\infty) = 1 \), the supremum and infemum of \( f \) are attained. Since \( 0 < f(r) < \infty \), it follows that \( \sup_{0 < r < \infty} f(r) < \infty \) and \( \inf f(r) > 0 \).

The proof for (2) is similar.