Problem 1: Consider the Hilbert space $H = \mathbb{C}^n$. Let $A \in \mathcal{B}(H)$, let $(e^{(j)})_{j=1}^n$ be the canonical basis, and let $A$ have the representation

$$
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
$$

in the canonical basis. We define the Hilbert-Schmidt norm of $A$ as

$$
||A||_{HS} = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.
$$

(a) Let $(\varphi^{(j)})_{j=1}^n$ be any ON-basis for $H$. Show that $||A||_{HS} = \sum_{j=1}^{n} ||A\varphi^{(j)}||^2$.

(b) Show that $||A|| \leq ||A||_{HS} \leq \sqrt{n}||A||$ for any $A \in \mathcal{B}(H)$.

(c) Find $G, H \in \mathcal{B}(H)$ such that $||G||_{HS} = ||G||$ and $||H||_{HS} = \sqrt{n}||H||$.

Problem 2: Let $H$ be a separable Hilbert space, and let $A \in \mathcal{B}(H)$. Suppose that $H$ has an ON-basis $(\varphi^{(j)})_{j=1}^{\infty}$ such that

$$
\sum_{j=1}^{\infty} ||A\varphi^{(j)}||^2 < \infty.
$$

Prove that if $(\psi^{(j)})_{j=1}^{\infty}$ is any other ON-basis, then

$$
\sum_{j=1}^{\infty} ||A\varphi^{(j)}||^2 = \sum_{j=1}^{\infty} ||A\psi^{(j)}||^2.
$$

Problem 3: Consider the linear space $L = \mathbb{R}^2$. Define for $x = (x_1, x_2) \in L$ the seminorms

$$
p_1(x) = |x_1|, \quad p_2(x) = |x_2|.
$$

Construct for $x \in L$, $j \in \{1, 2\}$, and $\varepsilon \in (0, \infty)$, the sets

$$
\mathcal{B}_{x,j,\varepsilon} = \{ y \in L : p_j(x - y) < \varepsilon \}.
$$

Describe these sets geometrically. What is the topology generated by the collection of semi-norms $\{p_1\}$? Is it Hausdorff? What is the topology generated by the collection of semi-norms $\{p_1, p_2\}$? Is it Hausdorff?