Problem 12.8: We want to prove that
\[ \|f - f_n\|_p^p = \int |f - f_n|^p \to \infty. \]
We know that \(|f - f_n|^p \to 0\) pointwise, so if we can only justify moving the limit inside the integral, we’ll be done.

First note that
\[ |f(x)| = \lim_{n \to \infty} |f_n(x)| \leq |g(x)|. \]
Then we can dominate the integrand as follows:
\[ |f - f_n|^p \leq (|f| + |f_n|)^p \leq (|g| + |g|)^p \leq 2^p|g|^p. \]
Since \(\int |g|^p < \infty\), we find that the Lebesgue dominated convergence theorem applies, and so
\[ \lim_{n \to \infty} \int |f - f_n|^p = \int |f - f|^p = \int 0 = 0. \]

Problem 12.16: Fix \(f \in L^p\) and \(\varepsilon > 0\). We want to prove that there exists a \(\delta > 0\) such that for \(|h| < \delta\), we have \(\|f - \tau_h f\|_p < \varepsilon\).

First pick \(\varphi \in C_c\) such that \(\|f - \varphi\|_p < \varepsilon/3\). Then
\[ \|f - \tau_h f\|_p \leq \|f - \varphi\|_p + \|\varphi - \tau_h \varphi\|_p + \|\tau_h \varphi - \tau_h f\|_p \]
\[ = \|f - \varphi\|_p + \|\varphi - \tau_h \varphi\|_p + \|\varphi - f\|_p < \varepsilon/3 + \|\varphi - \tau_h \varphi\|_p + \varepsilon/3. \]
Set \(R = \sup \{|x| : \varphi(x) \neq 0\}\). Since \(\varphi\) is uniformly continuous, there exists a \(\delta\) such that if \(|x - y| < \delta\), then \(|\varphi(x) - \varphi(y)| < \varepsilon/(3\mu(B_{R+1}(0)))^1/p\). Then, if \(h < \min(\delta, 1)\),
\[ \|\varphi - \tau_h \varphi\|_p = \int_{B_{R+1}(0)} |\varphi(x) - \varphi(x - h)|^p \, dx < \int_{B_{R+1}(0)} \frac{\varepsilon^p}{3p} \mu(B_{R+1}(0)) \, dx < \frac{\varepsilon^p}{3p}. \]

Problem 12.17: For \(n = 1, 2, 3, \ldots\), set \(I_n = (2^{-n}, 2^{-n+1})\), and \(f_n = 2^{n/p} \chi_{I_n}\). Then \(\|f_n\|_p = 1\) for all \(n\). Suppose \(m \neq n\), then
\[ \|f_n - f_m\|_\infty = 1, \]
and for \(p \in [1, \infty)\) we have
\[ \|f_n - f_m\|_p = \left(\int_0^1 (2^n \chi_{I_n} + 2^m \chi_{I_m})^p \, dx \right)^{1/p} = 2^{1/p}. \]
No subsequence of \((f_n)_{n=1}^\infty\) can be Cauchy, and therefore no subsequence can converge.

Problem 12.18: For \(n = 1, 2, 3, \ldots\), set \(I_n = (2^{-n}, 2^{-n+1})\), and \(f_n = 2^n \chi_{I_n}\). Let \((f_{n_j})_{j=1}^\infty\) be a subsequence of \((f_n)_{n=1}^\infty\). Define \(g \in L^\infty\) by
\[ g = \sum_{j=1}^\infty (-1)^j \chi_{I_{n_j}}, \]
and define \(\varphi \in (L^1)^*\) via \(\varphi(f) = \int f g\). Then \(\varphi(f_{n_j}) = (-1)^j\) (verify!) and so \((f_{n_j})\) cannot converge weakly. Since \(L^1\) is not reflexive, this does not contradict that Banach-Alaoglu theorem.
Problem 12.13: Set \( I = [0, 1] \) and let \( \Omega \) be a dense set in \( L^\infty(I) \). For \( r \in I \), set \( f_r = \chi_{[0, r]} \), and pick \( x_r \in \Omega \cap B_{1/3}(f_r) \). Since \( ||f_r - f_s|| = 1 \) if \( s \neq r \), we find that \( ||x_r - x_s|| \geq ||f_r - f_s|| - ||f_r - x_r|| - ||f_s - x_s|| \geq 1/3 \), so all the \( x_r \)'s are distinct. Therefore, \( \Omega \) must be uncountable, and \( L^\infty \) cannot be separable.

To prove that \( C(I) \) cannot be dense in \( L^\infty(I) \), simply note that if \( f = \chi_{[0,1/2]} \), and \( \varphi \in C(I) \), then

\[
||f - \varphi||_\infty \geq \max(||\varphi(1/2)||, ||1 - \varphi(1/2)||) \geq 1/2
\]

(verify this!).

An alternative argument for why \( C(I) \) cannot be dense in \( L^\infty(I) \): If \( \varphi_n \in C(I) \), and \( \varphi_n \to f \) in the supnorm, then \( (\varphi_n) \) is a Cauchy sequence with respect to the uniform norm (when applied to continuous functions, the uniform norm and the \( L^\infty \) norms are identical). Therefore, there exists a continuous function \( \varphi \) such that \( \varphi_n \to \varphi \) uniformly. Then \( f(x) = \varphi(x) \) almost everywhere. But not every equivalence class function in \( L^\infty \) has a continuous function in it (for instance \( f = \chi_{[0,1/2]} \)).

Problem 12.14: Let \( p \) and \( q \) be such that \( 1 \leq p < q \leq \infty \).

First we construct a function \( f \in L^p \setminus L^q \). Let \( \alpha \) be a non-negative number and set \( f(x) = x^\alpha \chi_{[0,1]} \).

Then

\[
||f||_p^p = \int_0^1 x^{-\alpha} dx,
\]

which is finite if \( \alpha p < 1 \). Moreover

\[
||f||_q^q = \int_0^1 x^{-\alpha} dx
\]

which is infinite if \( \alpha q > 1 \). Consequently, \( f \in L^p \setminus L^q \) if

\[
\frac{1}{q} < \alpha < \frac{1}{p}.
\]

To construct a function \( f \in L^q \setminus L^p \), set \( f = x^{-\alpha} \chi_{[1,\infty)} \).

Then

\[
||f||_p^p = \int_1^\infty x^{-\alpha} dx
\]

which is infinite if \( \alpha p < 1 \). Moreover

\[
||f||_q^q = \int_1^\infty x^{-\alpha} dx
\]

which is finite if \( \alpha q > 1 \). Thus, \( f \in L^1 \setminus L^p \) if

\[
\frac{1}{q} < \alpha < \frac{1}{p}.
\]

(The arguments above need slight modifications if \( q = \infty \), but the idea is the same.)

Consider the function

\[
f(x) = \frac{1}{(|x| (1 + \log^2 |x|))^{1/2}}.
\]

That \( f \in L^2 \) is clear, since

\[
||f||_2^2 = \int_{-\infty}^\infty \frac{1}{|x|(1 + \log^2 |x|)} dx = 2 \int_0^\infty \frac{1}{x(1 + \log^2 x)} dx = \{x = e^t\}
\]

\[
2 \int_{-\infty}^\infty \frac{1}{e^{t}(1 + t^2)} e^t dt = 2\pi.
\]
Moreover, if \( p > 2 \), then note that there exists a \( \delta > 0 \) such that
\[
x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1
\]
when \( x \in (0, \delta) \). Then
\[
\|f\|_p^p \geq \int_0^\delta \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} \, dx = \int_0^\delta \frac{1}{x^{(p-2)/2} (1 + \log^2 x)^{p/2}} \, dx = \infty.
\]

Analogously, if \( p < 2 \), then there exists an \( M \) such that
\[
x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1
\]
when \( x \geq M \). Then
\[
\|f\|_p^p \geq \int_M^\infty \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} \, dx = \int_M^\infty \frac{1}{x^{(p-2)/2} (1 + \log^2 x)^{p/2}} \, dx = \infty.
\]

**Problem 12.15:** Let \( \alpha \in (0, 1) \), and let \( m, n \in (1, \infty) \) be such that \( 1/m + 1/n = 1 \) (we will determine suitable values for \( \alpha, m, n \) later). Then from Hölder’s inequality we obtain
\[
(1) \quad \|f\|^r = \int |f|^r = \int |f|^{\alpha r} |f|^{(1-\alpha) r} \leq \left( \int |f|^{\alpha mr} \right)^{1/m} \left( \int |f|^{(1-\alpha) nr} \right)^{1/n}.
\]

In order to obtain the desired right hand side, we must pick \( \alpha, m, n \) so that
\[
\alpha mr = p,
\]
\[
(1-\alpha) nr = q,
\]
\[
(1/m) + (1/n) = 1.
\]

To obtain an equation for \( \alpha \), we eliminate \( m \) and \( n \):
\[
\frac{(1-\alpha)r}{q} = \frac{1}{n} = 1 - \frac{1}{m} = 1 - \frac{\alpha r}{p}.
\]

Solving for \( \alpha \) we obtain
\[
\alpha = \frac{pq - pr}{rq - rp} = \frac{1/r - 1/q}{1/p - 1/q}.
\]

Equation (1) now takes the form
\[
\|f\|^r \leq \left( \left( \|f\|^p \right)^{1/m} \left( \|f\|^q \right)^{1/n} \right)^{1/r} = \|f\|_{p/mr}^{p/mr} \|f\|_{q/nr}^{q/nr}.
\]

Finally note that
\[
\frac{p}{mr} = \alpha = \frac{1/r - 1/q}{1/p - 1/q},
\]
\[
\frac{q}{nr} = 1 - \alpha = 1 - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/r}{1/p - 1/q}.
\]
**Problem 1:** Let $\lambda$ be a real number such that $\lambda \in (0, 1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda) b,$$

with equality iff $a = b$.

**Solution:** For $b = 0$ equation (2) reduces to $0 \leq \lambda a$ which is clearly true.

When $b \neq 0$ we divide (2) by $b$ and set $t = a/b$ to obtain

$$t^\lambda \leq \lambda t + 1 - \lambda.$$

Set $f(t) = \lambda t + 1 - \lambda - t^\lambda$.

We need to prove that $f(t) \geq 0$ when $t \geq 0$. First note that $f(0) = 1 - \lambda > 0$ and that $\lim_{t \to \infty} f(t) = \infty$. Since $f$ is differentiable, we therefore need only investigate the points where $f'(t) = 0$. We find

$$f'(t) = \lambda - \lambda t^\lambda - 1$$

so $f'(t) = 0$ happens only when $t = 1$. Now $f(1) = 0$ so it follows that $f(t) \geq 0$ for all $t \geq 0$, and that $f(t) = 0$ iff $t = 1$ (which is to say $a = b$).

**Problem 2:** [Hölder’s inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $(X, \mu)$ be a measure space, and suppose that $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Prove that equality holds iff $\alpha |f|^p = \beta |g|^q$ for some $\alpha, \beta$ such that $\alpha \beta \neq 1$.

**Solution:** Suppose $\|f\|_p = 0$, then $f = 0$ a.e. and so (3) holds since both sides are identically zero. Analogously, (3) holds when $\|g\|_q = 0$.

Now suppose $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. Set

$$a = \frac{\|f(x)\|^p}{\|f\|_p}, \quad b = \frac{|g(x)|^q}{\|g\|_q}, \quad \lambda = \frac{1}{p}.$$

Then invoke (2), observing that $q(1 - \lambda) = q(1 - 1/p) = q(1/q) = 1$, to obtain

$$\frac{|f(x)| |g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \left(1 - \frac{1}{p}\right) \frac{|g(x)|^q}{\|g\|_q^q}.$$

Integrate over $X$ to obtain

$$\frac{1}{\|f\|_p \|g\|_q} \int_X |f(x)| |g(x)| \, d\mu(x) \leq \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \left(1 - \frac{1}{p}\right) \frac{\|g\|_q^q}{\|g\|_q^q}.$$

Multiply by $\|f\|_p \|g\|_q$ to obtain (3).
**Problem 3:** [Minkowski’s inequality] Let \((X, \mu)\) be a measure space, and let \(p\) be a real number such that \(1 \leq p \leq \infty\). Prove that for \(f, g \in L^p(X, \mu)\),

\[ ||f + g||_p \leq ||f||_p + ||g||_p. \]

**Solution:**

Suppose \(p = 1\):

\[ ||f + g||_1 = \int |f(x) + g(x)| \leq \int (|f(x)| + |g(x)|) = \int |f(x)| + \int |g(x)| = ||f||_1 + ||g||_1. \]

Suppose \(p = \infty\):

\[ ||f + g||_\infty = \text{ess sup } |f(x) + g(x)| \leq \text{ess sup}(|f(x)| + |g(x)|) \]

\[ \leq \text{ess sup } |f(x)| + \text{ess sup } |g(x)| = ||f||_\infty + ||g||_\infty. \]

Suppose \(p \in (1, \infty)\): The triangle inequality yields

\[ |f(x) + g(x)|^p = |f(x) + g(x)||f(x) + g(x)|^{p-1} \leq (|f(x)| + |g(x)|)|f(x) + g(x)|^{p-1}. \]

Integrate both sides:

\[ ||f + g||_p^p \leq \int |f(x)||f(x) + g(x)|^{p-1} + \int |g(x)||f(x) + g(x)|^{p-1}. \]

Now apply Hölder:

\[ ||f + g||_p^p \leq ||f||_p ||f + g||^{p-1}_q + ||g||_p ||f + g||^{p-1}_q = (||f||_p + ||g||_p) \left( \int |f(x) + g(x)|^{q(p-1)} \right)^{1/q}. \]

Now use that \(q = 1/(1 - 1/p) = p/(p - 1)\) to see that \(q(p - 1) = p\) to get

\[ ||f + g||_p^p \leq (||f||_p + ||g||_p) \left( \int |f(x) + g(x)|^{p} \right)^{1/q} = (||f||_p + ||g||_p) ||f + g||_p^{p/q}. \]

Observe that \(p/q = p(1 - 1/p) = p - 1\) to obtain

\[ ||f + g||_p^p \leq (||f||_p + ||g||_p) ||f + g||_p^{p - 1} \]

which gives Minkowski upon division by \(||f + g||_p^{p - 1}\).