The Sobolev Spaces $H^k(\mathbb{T})$

Set $C^k(\mathbb{T})$ = Space of Functions on $\mathbb{T}$ s.t. $\hat{f}_{(j)} \in C(\mathbb{T})$ for $j=0,1,2,...,k$

$\|f\|_{C^k} = \bigg\| \sum_{j=0}^{k} \|f_j\|_u \bigg\|

Suppose that $f \in C^1(\mathbb{T})$, and that $f = \sum \lambda_n e_n$, $f' = \sum \lambda_n' e_n$.

Then $\lambda_n = \int e^{-inx} f'(x) dx = \int e^{-inx} \frac{d}{dx} \sum \lambda_n e_n dx = \int \frac{d}{dx} e^{-inx} \sum \lambda_n e_n dx = \Pi \alpha_n$

So if $f$ has Fourier series $(\alpha_n)$, then $f'$ has F-series $(i\alpha_n)$.

If $f = \sum \lambda_n e_n$ be a function in $L^2(\mathbb{T}) \setminus C^1(\mathbb{T})$ for which $\sum \lambda_n^2 |\alpha_n|^2 < \infty$, then we define the weak derivative of $f$ by

$f' = \sum i \alpha_n e_n$

We set $H^1(\mathbb{T}) = \{ f \in L^2 \mid f, f' \text{ are measurable, and } \sum \lambda_n^2 |\alpha_n|^2 < \infty \}$

and $\langle f, g \rangle_{H^1} = \sum (1 + |\alpha_n|^2) \lambda_n \alpha_n$, where $f = \sum \lambda_n e_n$, $g = \sum \mu_n e_n$.

By Parseval: $\langle f, g \rangle_{H^1} = \sum \lambda_n \alpha_n + \sum |\alpha_n| |\lambda_n|$

$f = \sum \lambda_n e_n$, $f' = \sum i \lambda_n e_n$, $\overline{f'} = \sum \lambda_n \alpha_n e_n$

Moreover, $\int f(x) g(x) dx = \langle f, g \rangle = \sum \lambda_n \alpha_n = -\sum \lambda_n \mu_n = -\sum \lambda_n (i\alpha_n) = \int f'(x) g(x) dx = \langle f', g \rangle = \sum \lambda_n |\alpha_n| = \langle \overline{f'}, g \rangle = -\int \overline{f(x)} g'(x) dx$

This is integration by parts!

(The boundary terms vanish thanks to periodicity.)
Aside: Weak derivatives can be defined without Fourier methods.

Suppose that \( f \in L^2(I) \) is a function s.t.
\[
\left| \int_I \overline{\phi'(x)} \phi(x) \, dx \right| \leq M \| \phi \|_2 \quad \forall \phi \in C^\infty(I).
\]
Then the map \( F^* \) is a bounded linear functional defined on \( C^\infty(I) \), which is a dense subset of \( L^2(I) \) (since \( \Phi \subset C^\infty(I) \)).

It can be extended to a map \( F \in L^2(I)^* \).

By the Riesz rep\^* theorem, \( \exists! h \in L^2(I) \) s.t.
\[
F(\phi) = < h, \phi > \quad \forall \phi \in L^2(I).
\]
This \( h \) is the weak derivative of \( f \).

Note that \[
\forall \phi \in C^\infty(I),\quad < h, \phi > = \int_I h(x) \phi(x) \, dx = -< f, \phi' >.
\]

End of aside

More generally, define for \( k \geq 0 \):
\[
H^k(I) = \text{The space of all functions } f = \sum a_n e_n \text{ for which } \sum (1 + |n|^2)^k |a_n|^2 < \infty.
\]

For \( f, g \in H^k \), set \[
< f, g >_{H^k} = \sum (1 + |n|^2)^k \overline{a_n} b_n = \sum \overline{a_n} b_n = \int_I f(x) \overline{g(x)} \, dx.
\]

Lemma: Suppose that \( \sum f_n = \sum \alpha_n e_n = \sum_{n=0}^{\infty} \alpha_n e_n \in L^2(I) \) for some \( k \geq 0 \).

Set \( \sum a_n = \sum \alpha_n e_n \). Then \( f \in C^\infty(I) \) s.t.
\[
|f|_r = \| f \|_{L^2}.
\]
Lemma: Suppose that $k > \frac{1}{2}$. Then if $C_k < \infty$ then the following proposition holds:

For every $f \in H^k(I)$, we have

$$\|f - f_N\|_u \leq \frac{C_k}{N^{k-\frac{1}{2}}} \|f\|_{H^k}$$

where $f_N = \sum_{n=0}^{N} a_n e_n$

In other words, the lemma asserts that if $f \in H^k$ for $k > \frac{1}{2}$, then the Fourier series converges uniformly to $f$.

Since each $f_N$ is continuous, this proves that $f \in C(I)$.

(And so $u \in H^k(I) \subset C(I)$.) This is a special case of

Proof: First we prove that $\left( f_N \right)_N$ is a Cauchy sequence in $C(I)$.

If $N < M$, then

$$\|f_N - f_M\|_u = \sup_x \left| \sum_{n=0}^{N} a_n \frac{e^{inx}}{\sqrt{2\pi}} - \sum_{n=0}^{M} a_n \frac{e^{inx}}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2\pi}} \sum_{j=M+1}^{\infty} \log j \leq \frac{1}{\sqrt{2\pi}} \sum_{j=M+1}^{\infty} \frac{1}{j} \log j$$

$$\leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1(I)} \left( \int_{N}^{\infty} \frac{1}{j^{2k-1}} \, dj \right)^{1/2} \left( \frac{1}{\sqrt{2\pi}} \right)^{1/2} \|f\|_{H^k}$$

$$\leq \frac{1}{\sqrt{2\pi}} \|f\|_{H^k} \left( \int_{N}^{\infty} \frac{1}{j^{2k-1}} \, dj \right)^{1/2} \|f\|_{H^k} \left( \left[ \int_{N}^{\infty} \frac{1}{j^{2k-1}} \, dj \right]_{N}^{\infty} \right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}} \|f\|_{H^k} \frac{1}{\sqrt{2\pi} N^{k-1/2}} N^{k-1/2}$$

Set $C_k = \frac{1}{\sqrt{2\pi} N^{k-1/2}}$.

Since $(C(I))$ is complete, there exists $f \in C(I)$ such that $f_N$ converges uniformly.

But then $f_N \rightharpoonup f$ in $L^2$ as well, so we must have $f = f$.

Finally, note that

$$\|f - f_N\|_u \leq \lim_{N \to \infty} \|f_N - f_M\|_u \leq \frac{C_k}{N^{k-\frac{1}{2}}} \|f\|_{H^k}$$
We proved that $H^k(T^d) \subseteq C(T^d)$ when $k > \frac{1}{2}$.

More generally we have

\textbf{Thm} Sobolev embedding

Suppose that $d$ is a positive integer, and that $k > \frac{d}{2}$. Then $H^k(T^d) \subseteq C(T^d)$.

Moreover, the map

$$E: H^k(T^d) \to C(T^d): f \mapsto f$$

is compact.
Let us start with a brief review of techniques for solving linear systems of ODEs.

Let $A$ be a symmetric $N \times N$ matrix and consider the eq

\[
\begin{aligned}
&Au(t) = \frac{d}{dt} u(t) \\
&u(0) = f
\end{aligned}
\]

for the vector-valued function $u = u(t) \in \mathbb{R}^N$.

Spectral theorem: There is an ON-basis $(\phi_n)_{n=1}^N$ s.t.

\[
A\phi_n = \lambda_n \phi_n
\]

for some $\lambda_n \in \mathbb{R}$.

Set $u(t) = \sum_{n=1}^N \alpha_n(t) \phi_n$ for some functions $\alpha_n(t)$ to be determined.

We find

\[
\begin{aligned}
Au &= \sum_{n=1}^N \alpha_n(t) A\phi_n = \sum_{n=1}^N \alpha_n(t) \lambda_n \phi_n \\
\frac{d}{dt} u &= \sum_{n=1}^N \alpha_n'(t) \phi_n
\end{aligned}
\]

\[
\Rightarrow \alpha_n'(t) = \lambda_n \alpha_n(t)
\]

\[
\Rightarrow \alpha_n(t) = c_n e^{\lambda_n t}
\]

for $n=1,2,\ldots,N$.

We see that $\alpha_n(t) = c_n e^{\lambda_n t}$ for some numbers $(c_n)$.

To determine $c_n$, use the initial condition:

\[
\left. u(t) \right|_{t=0} = \sum_{n=1}^N c_n \phi_n = f
\]

\[
\Rightarrow c_n = \langle \phi_n, f \rangle
\]

(since $(\phi_n)_{n=1}^N$ is an ON-basis.)

We find that the unique solution is

\[
\begin{aligned}
&u(t) = \sum_{n=1}^N \langle \phi_n, f \rangle e^{\lambda_n t} \phi_n \\
&= e^{At} f \quad \text{by def. of the matrix exponential.}
\end{aligned}
\]
Now let us try to emulate the same technique for the "heat equation". First we do it heuristically!

Set $I = [0, a]$ and let $u = u(t, x)$ be an unknown function. The heat eq. reads

$$\frac{d^2 u}{dx^2} = \frac{d}{dt} u \quad \text{for } t > 0, \ x \in I$$

- $u(t, 0) = 0$ \quad \text{homogeneous boundary conditions}
- $u(t, a) = 0$
- $u(0, x) = f(x)$ \quad \text{initial condition}

Set $A = \frac{d^2}{dx^2}$ and $\phi_n(x) = \sin(nx)$. Evens of $\frac{d^2}{dx^2}$.

Observe that $A\phi_n = -n^2 \phi_n$.

Set $\lambda_n = -n^2$

Now make the Ansatz $u(x, t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin(nx)$.

$$U_{xx} = \sum_{n=1}^{\infty} -n^2 \alpha_n(t) \sin(nx) \int \Rightarrow \alpha_n(t) = C_1 e^{-n^2 t}$$

$$U_t = \sum_{n=1}^{\infty} \alpha_n'(t) \sin(nx)$$

Initial cond. $: \sum_{n=1}^{\infty} C_n \sin(nx) = f(x) \Rightarrow C_n = \frac{2}{a} \int_0^a f(x) \sin(nx) dx$

So $u(x, t) = \sum_{n=1}^{\infty} \frac{\langle \phi_n, f \rangle}{\| \phi_n \|^2} e^{-n^2 t} \sin(nx)$

$$= e^{At} f \quad \text{by defn}!$$
Formalize the solution of heat eqn using Fourier methods.

For \( t > 0 \), let \( \varphi(t, \cdot) \) be a function on \( T \) in the terms

\[
\begin{align*}
U_{xx} &= U_t \\
U(0, x) &= \varphi(x)
\end{align*}
\]

Assume for now & that \( \phi \in \mathcal{D} \) the trigonometric polynomials.

If \( \phi = \sum_{n=-N}^{N} a_n e^{inx} \), so that \( \varphi(x, t) = \sum_{n=-N}^{N} \frac{a_n}{\alpha_n} e^{-\alpha_n t} \phi_n \)

where \( \phi_n(x) = e^{inx} \). This is a classical solution.

Define \( T(t) : \mathcal{D} \to L^2 \) by

\[
|| T(t) \phi ||^2_{L^2} = \sum_{n=-N}^{N} \frac{a_n^2 e^{-\alpha_n t}}{\alpha_n} \leq \sum_{n=-N}^{N} a_n^2 \Rightarrow || T(t) || \leq 1
\]

Since \( T(t) \) is cont & \( \mathcal{D} \) is dense, \( T(t) \) can be extended to all of \( L^2(T) \).

Properties of \( T(t) \):

* \( T(0) = I \)
* \( T(t)T(s) = T(t+s) \) for \( s, t > 0 \)
* \( T(t) \to I \) strongly as \( t \to 0 \).

Note that \( T(t) \) does not converge in norm at \( t > 0 \). We call \( (T(t))_{t>0} \) a strongly continuous semigroup.

For \( t > 0 \), \( T(t) \phi \) is a classical solution of \( \text{PDE} \).

To see this, we prove that \( T(t) \phi \in C^\infty \).

Fix \( n \), then \( T(t) \phi \in H^n(\mathbb{R}) \) since

\[
|| T(t) \phi ||^2_{H^n} = \sum_{n=1}^{\infty} n^2 |a_n|^2 e^{-2\alpha_n t} \leq \sum_{n=1}^{\infty} |a_n|^2 = || \phi ||^2_{L^2}
\]

Thus \( T(t) \phi \in C^{\infty}(\mathbb{R}) \) for all \( n \).
We have constructed one solution of PDE. It remains to prove uniqueness.

\[ U_{xx} = U_t \Rightarrow \int_0^\infty U_t U_{xx} = \int_0^\infty u x \, dt = -\int_0^\infty U_t \, dt = \frac{1}{2} \int_0^\infty U^2 \, dt = \frac{1}{2} \|
abla U\|^2. \]

Thus \( \|u(t)\|_2 \) is non-increasing as \( t \to \infty \). \( \Rightarrow \|u(0)\|_2 \leq \|u(t)\|_2 \).

Now assume \( u \) & \( v \) both solve PDE and set \( w = u - v \).

Then \( w_{xx} = w_t \), \( w(x,0) = 0 \) \( \Rightarrow \|w\|_2 \leq \|v\|_2 = 0. \)
**PROJECTIONS ON LINEAR SPACES (NO TOPOLOGY)**

**Def.** Let \( X \) be a linear space. An operator \( P: X \to X \) is a \textit{proj} if \( P^2 = P \).

**Lemma 1.** Let \( P \) be a \textit{proj} on a linear space \( X \).

Set \( M = \text{ran} \, P, \ N = \text{ker} \, P \). Then:

(i) \( M = \{ x \in X : x = Px \} \)

(ii) \( X = \text{span} (M, N) \)

(iii) \( M \cap N = \{ 0 \} \) \( \implies \) \( X = \text{ran} \, P \oplus \text{ker} \, P \)

**Proof.**

(i) Set \( A = \{ x : x = Px \} \).

Obviously, \( A \subseteq M \).

Conversely, assume \( x \in M \), then \( J \) s.t. \( x = P_J \) \( \implies \) \( P_J = P_J = x \).

(ii) Given \( x \), set \( J = P_X \) & \( z = x - P_J \).

Then \( x = y + z \), \( y \in M \) & \( Pz = P_x - P_J = P_x - P_J = 0 \) so \( z \notin N \).

(iii) Suppose \( x \in M \cap N \). Then \( x = Px = 0 \).

**Lemma 2.** Suppose that \( X \) is a linear space, \( x \in M \), \( x \in N \).

Then \( x \in \text{proj} \, P \) s.t. \( \text{ran} \, P = M, \ \text{ker} \, P = N \).

**PROJECTIONS ON BANACH SPACES**

**Def.** Let \( X \) be a Banach space.

A map \( P: X \to X \) is a \textit{proj} if \( P^2 = P \) & \( \| P \| < \infty \).

**Lemma 1.** Let \( P \) be a \textit{proj} on a Banach space \( X \).

Set \( M = \text{ran} \, P \) & \( N = \text{ker} \, P \). Then:

(i) \( M = \{ x : x = Px \} \)

(ii) \( X = \text{span} (M, N) \)

(iii) \( M \cap N = \{ 0 \} \)

(iv) \( M \) & \( N \) are closed.

**Proof.**

(iv) \( M = \text{ker} \, P \) is closed since \( P \) is cont.

\( N = \text{ker} \, (I-P) \) is closed.
Lemma 2 Let $X$ be a Banach space, and let $M, N$ be closed linear subspaces s.t. $X = M \oplus N$.

Then if a proj $P$ s.t. $M = \text{ran } P$, $N = \text{ker } P$.

Proof: We only need to prove that $P$ is continuous.

Step 1 Let $X$ denote the space $X$ equipped with the norm $\|x\| = \|y\| + \|z\|$. It is simple to prove that $\| \cdot \|$ is a norm.

That $X$ is complete follows from the closedness of $M \oplus N$.

Step 2 Note that $P \in B(X, X)$ since $\|Px\| = \|y\| + \|z\| = \|y\| + \|z\| = \|x\|$.

Step 3 Prove that $X = X$.

That $P \in B(X)$ follows if we can prove that $X = X$ are homeomorphic.

Consider the embedding map $J: X \to X$.

$J$ is obviously bijective.

$J$ is continuous since $\|x\| = \|y + z\| = \|y\| + \|z\| = \|x\|$.

That $J^*$ is cont. now follows from the open mapping theorem.

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**PROJECTIONS ON HILBERT SPACES.**

Def: the same as Banach space.

Lemma 1 is identical to Banach space case.

Lemma 2 Let $H$ be a H.S. and let $M$ be a closed linear subspace.

Then a proj $P$ s.t. $\text{ran } P = M$, $H = M \oplus \text{ker } P$.

Proof We proved previously that $H = M \oplus M^\perp$.

Now apply the Banach space lemma 2.

Given $x \in H$, we have $x = y + z$. Set $Px = y$ where $y \in M$.

$\|Pz\| = \|y\| = \sqrt{\|x\|^2 - \|z\|^2} < \|x\|$, so $P$ is cont.
Def. Let $P$ be a proj on a H.S. $H$.
If $\text{ran} \ P \subseteq \ker P$, we say that $P$ is an orthogonal proj.

Lemma. Let $H$ be a H.S. and let $P$ be a proj on $H$.

TFAE:
1. $P$ is orthogonal
2. $\langle Px, y \rangle = \langle x, Py \rangle \quad \forall x, y \in H$
3. $\|P\| = 0$ (if $P$ is not zero)

Proof. Homework

Geometry:

$Px = y$.

If $N$ is not perpendicular to $N$, then $\|Nx\| \neq \|x\|$ and so $\|P\| > 1$.

$\exists x$ s.t.

$\|P\| \leq \frac{1}{\cos \theta}$

In a Banach space, it is possible for $\|P\| = \infty$. 