Note: You may want to save problems marked with a star for last.

Problem 1: (15 points) Let \( g, h \in \mathcal{L}^2(\mathbb{R}) \) and set \( f = g * h \). Prove that \( ||f||_u \leq ||g||_u ||h||_u \) (where \( ||f||_u = \sup_x |f(x)| \)). Is it necessarily the case that \( f \in \mathcal{C}_0(\mathbb{R}) \)? Motivate your answer briefly.

Problem 2: (26 points) In this problem, \( S = \mathcal{S}(\mathbb{R}) \) is the Schwartz space over the real line, \( a \) is a non-zero real number, and \( \mathcal{F} \) is the Fourier transform.

(a) [6p] Define the operator \( D_a : S \rightarrow S \) via \( [D_a \varphi](x) = \varphi(a x) \). Show that for some \( b, c \in \mathbb{R} \) \( \mathcal{F} D_a \varphi = bym D_c \mathcal{F} \varphi \).

(b) [6p] State the appropriate definition of the operator \( D_a : S^* \rightarrow S^* \), and derive for \( T \in S^* \) a formula for \( \mathcal{F} D_a T \) analogous to (EQ1). Be careful in motivating your work!

(c) [6p] Fix a function \( h \in \mathcal{C}_0(\mathbb{R}) \) (i.e. \( h \) is bounded and continuous), and set \( f_n = D_{1/n} h \) for \( n = 1, 2, 3, \ldots \). Prove that the sequence \( (f_n)_{n=1}^\infty \) converges in \( S^* \) and give the limit.

(d) [6p] With \( f_n \) as in (c), set \( \hat{f}_n = \mathcal{F} f_n \). Does the sequence \( (\hat{f}_n)_{n=1}^\infty \) converge in \( S^* \)? If so, to what?

(e*) [2p] Give an example of a distribution \( h \in \mathcal{S}^* \) such that \( (D_{1/n} h)_{n=1}^\infty \) does not converge in \( \mathcal{S}^* \).

Problem 3: (25 points)

(a) [5p] For \( d \) a positive integer, and \( s \) a real number, define the Sobolev space \( H^s(\mathbb{R}^d) \).

(b) [5p] For which \( s \), if any, is it necessarily the case that all functions in \( H^s(\mathbb{R}^d) \) are continuous?

(c) [10p] Let \( f \in \mathcal{L}^2(\mathbb{R}) \). Show that the equation \( -u'' + u = f \) has a unique solution \( u \in H^2(\mathbb{R}) \).

(d*) [5p] Give an example of a function \( f \in \mathcal{L}^2(\mathbb{R}^2) \) such that the equation
\[
- \frac{\partial^2 u}{\partial x_1^2} + u = f,
\]
does not have a solution in \( H^2(\mathbb{R}^2) \).

Problem 4: (12 points)

(a) [4p] State the definition of a measure.

(b) [4p] Let \( (X, \mathcal{T}) \) be a topological space. State the definition of the Borel \( \sigma \)-algebra associated with \( (X, \mathcal{T}) \).

(c) [4p] State the definition of the essential supremum.

Problem 5: (12 points) Let \( \mathbb{N} \) denote the set of positive integers, and let \( \mathcal{A} \) denote the collection of all subsets of \( \mathbb{N} \). Let \( (\alpha_n)_{n=1}^\infty \) be a sequence of real numbers, and define a function
\[
\mu : \mathcal{A} \rightarrow \mathbb{R} : \Omega \mapsto \sum_{n \in \Omega} \alpha_n.
\]
Under what conditions on the numbers \( (\alpha_n) \) is \( \mu \) a measure? Is it ever a finite measure? Is it ever a \( \sigma \)-finite measure? No motivation required.