Problem 1: (30p total, 5p per question) Let $H$ denote a Hilbert space with an ON-basis $(e_n)_{n=1}^{\infty}$.
Which of the following statements are necessarily true? No motivation required.

(a) $e_n \rightarrow 0$.

(b) Suppose that $x, x_n \in H$ and $\lim_{n \rightarrow \infty} (x_n, e_m) = (x, e_m)$ for every $m$. Then $x_n \rightarrow x$.

(c) Suppose that $P \in B(H)$ is such that $P^2 = P$ and $P \neq 0$. Then $||P|| = 1$ if and only if $P^* = P$.

(d) Suppose $A \in B(H)$ is self-adjoint. Then $C = \exp(iA)$ is unitary.

(e) Suppose that $A, B \in B(H)$, that $A$ is coercive, and that $B$ is positive. Then $A + B$ is coercive.

(f) Suppose that $A, B \in B(H)$, and that $A$ is self-adjoint. Then $E = B A B^*$ is self-adjoint.

Problem 2: (26p) Let $T$ denote the one-dimensional torus, parameterized with the interval $I = (-\pi, \pi]$. Set $e_n(x) = e^{inx}/\sqrt{2\pi}$, and let $P$ denote the set of all finite linear combinations of basis functions $e_n$, as usual. Let $z$ denote a non-zero complex number and consider the PDE

\[ \frac{\partial u}{\partial t} = z \frac{\partial^2 u}{\partial x^2}, \]

along with periodic boundary conditions, and with the initial condition

\[ u(x, 0) = f(x), \quad x \in I. \]

(a) (10p) Construct the solution operator $T(t) : P \rightarrow P$ that maps a function $f \in P$ to a function $u = T(t) f$ that solves (1) and (2).

(b) (8p) Suppose that $t > 0$. For which values of $z$ can the solution operator $T(t)$ be extended to a bounded operator on $L^2(T)$? (Recall that $P$ is dense in $L^2(T)$.)

(c) (8p) Suppose that $t > 0$ and that $z$ is such that $T(t)$ is a bounded operator on $L^2(T)$. Suppose that $f \in L^2(T)$. For which values of $z$ can you guarantee that $T(t) f \in C^1(T)$? Can you ever guarantee that $T(t) f \in C^2(T)$?

Problem 3: (24p) Let $H$ denote a Hilbert space.

(a) (8p) Suppose that $U, T \in B(H)$, that $U$ is unitary, and that $||T|| = 1/3$. Prove that $A = U + T$ is continuously invertible.

(b) (8p) Suppose that $S \in B(H)$ and that $S$ is skew-symmetric. Prove that $\text{ran}(I + S)$ is closed.

(c) (8p) For the particular case of $H = L^2(I)$ with $I = [-1, 1]$, give an example of a unitary operator $U \in B(H)$ and a skew-symmetric operator $S \in B(H)$ such that $\text{ran}(U + S)$ is not closed.

Problem 4: (20p) Recall that if $A$ is an $n \times n$ matrix with complex entries, then

\[ \text{ran}(A) = (\ker(A^*))^{-1}. \]

Now suppose that $H$ is a Hilbert space, and $A \in B(H)$. State and prove a relationship analogous to (3) that $A$ must satisfy.