Homework set 8 — APPM5450, Spring 2010

From the textbook: 11.1(b,c), 11.2, 11.6, 11.7, 11.8, 11.10.

Problem 2: Prove that if \( f \in C^\infty(\mathbb{R}^d) \), and for every \( \alpha \in \mathbb{Z}^d \), there exist finite \( C \) and \( N \) such that \( |\partial^\alpha f(x)| \leq C(1 + |x|^N) \), then \( f\varphi \in S \) whenever \( \varphi \in S \). Moreover, prove that if \( \varphi_n \to \varphi \) in \( S \), then \( f\varphi_n \to f\varphi \) in \( S \).

Problem 3: Demonstrate that a tempered function is not necessarily of at most polynomial growth by constructing a continuous function \( f \) on \( \mathbb{R} \) such that

\[
(1) \quad \int_{-\infty}^{\infty} |f(x)| \, dx < \infty,
\]

but

\[
(2) \quad \sup_{x \in \mathbb{R}} \frac{|f(x)|}{(1 + |x|^2)^{k/2}} = \infty, \quad \forall \ k \in \{0, 1, 2, \ldots\}.
\]

If you would like to make the problem slightly harder, then construct a function \( f \) that satisfies (2) and also

\[
(3) \quad \int_{-\infty}^{\infty} (1 + |x|^2)^{k/2} |f(x)| \, dx < \infty, \quad \forall \ k \in \{0, 1, 2, \ldots\}.
\]

Problem 4 (optional): Let \( \mathcal{D} \) denote the linear space \( C^\infty_c(\mathbb{R}^d) \). We define a topology on \( \mathcal{D} \) by saying that \( \varphi_n \to \varphi \) if and only if there exists a compact set \( K \subseteq \mathbb{R}^d \) such that \( \text{supp}(\varphi_n) \subseteq K \) for all \( n \), and \( ||\partial^\alpha \varphi_n - \partial^\alpha \varphi||_u \to 0 \) for all \( \alpha \in \mathbb{Z}_+^d \).

(a) Prove that \( \mathcal{D} \) is a linear subspace of \( S \).
(b) Prove that the set \( \mathcal{D} \) is not closed in the topology of \( S \).
(c) Prove that if \( \varphi_n \to \varphi \) in \( \mathcal{D} \), then \( \varphi_n \to \varphi \) in \( S \).