Exercise 7.13: Set $I = [0, 1]$ and consider the equation

$$i u_t = -u_{xx}, \quad x \in I, \quad t > 0,$$

for a complex valued function $u = u(x, t)$ with homogeneous boundary conditions,

$$u(0, t) = u(1, t) = 0,$$

and initial condition

$$u(x, 0) = f(x).$$

Set

$$e_n(x) = \sqrt{2} \sin(nx).$$

Then $(e_n)_{n=1}^\infty$ forms an ON-basis for $L^2(I)$. We look for a solution

$$u(x, t) = \sum_{n=1}^\infty \alpha_n(t) e_n(x).$$

Inserting (4) into (1) and (3), we find that

$$i \alpha'_n = n^2 \alpha_n, \quad \alpha_n(0) = f_n,$$

where $f_n = (e_n, f)$. The solution is

$$\alpha_n(t) = f_n e^{-i n^2 t}.$$

Since $|\alpha_n(t)| = |f_n|$ for any $t$, it follows directly from Parseval that

$$||u(t)||^2_{L^2(I)} = \sum_{n=1}^\infty |\alpha_n(t)|^2 = \sum_{n=1}^\infty |f_n|^2 = ||f||^2,$$

and that (using that the cosines also form an ON-set)

$$||u_x(t)||^2_{L^2(I)} = ||\sum_{n=1}^\infty f_n e^{-i n^2 t} n \sqrt{2} \cos(nx)||^2_{L^2(I)} = \sum_{n=1}^\infty |n f_n|^2 = ||f_x||^2.$$

For a direct proof, set $v = \text{Re}(u)$ and $w = \text{Im}(u)$ so that $u = v + i w$. Then (1) takes the form

$$v_t = -w_{xx} \quad w_t = v_{xx}.$$

Now

$$\frac{d}{dt} \int_0^1 |u|^2 \, dx = \frac{d}{dt} \int_0^1 (v^2 + w^2) \, dx = 2 \int_0^1 (v_t v + w_t w) \, dx$$

$$= 2 \int_0^1 (-w_{xx} v + v_{xx} w) \, dx = 2 \int_0^1 (w_x v_x - v_x w_x) \, dx = 0.$$

The second to last step was partial integration where the boundary terms vanish due to (2). Analogously,

$$\frac{d}{dt} \int_0^1 |u_x|^2 \, dx = \frac{d}{dt} \int_0^1 (v_x^2 + w_x^2) \, dx = 2 \int_0^1 (v_{xt} v_x + w_{xt} w_x) \, dx$$

$$= 2 \int_0^1 (-v_t v_{xx} - w_t w_{xx}) \, dx = 2 \int_0^1 (-v_t w_x + w_t v_x) \, dx = 0.$$

In the second calculation we used differentiation, (2) takes the form

$$v_t(0, t) = v_t(1, t) = w_t(0, t) = w_t(1, t) = 0, \quad t > 0.$$
Exercise 8.3: Let $P$ and $Q$ be orthogonal projections. Set $M = \text{ran}(P)$ and $N = \text{ran}(Q)$. TFAE:

(a) $M \subseteq N$
(b) $QP = P$
(c) $PQ = P$
(d) $\|Px\| \leq \|Qx\| \quad \forall x$
(e) $(x, Px) \leq (x, Qx) \quad \forall x$

Proof:

(a) $\Rightarrow$ (b): Assume $M \subseteq N$. Then for any $x$, $Px \in M \subseteq N$, so $QPx = Px$.

(b) $\Rightarrow$ (a): Assume $QP = P$. Pick $y \in M$. Then $y = Px$ for some $x$. Then $Qy = QPx = Px = y$ so $y \in N$.

(a) $\Leftrightarrow$ (c):

$$M \subseteq N \quad \Leftrightarrow \quad N^\perp \subseteq M^\perp$$

$$\Leftrightarrow \quad Py = 0 \quad \forall y \in N^\perp$$

$$\Leftrightarrow \quad P(I - Q)x = 0 \quad \forall x$$

$$\Leftrightarrow \quad P = PQ$$

(c) $\Rightarrow$ (d): Assume $PQ = P$. Since $\|P\| \leq 1$ we have $\|Px\| = \|PQx\| \leq \|Qx\|$ for any $x$.

(d) $\Rightarrow$ (a): Assume that (a) is false. Then there is an $x \in M \setminus N$. Since $x \in M$ we have $x = Px$ and so

$\|Px\|^2 = \|x\|^2 = \|Qx + (I - Q)x\|^2 = \|Qx\|^2 + \|(I - Q)x\|^2$.

Now observe that $\|(I - Q)x\| > 0$ since $x \notin N$. Consequently,

$\|Qx\|^2 = \|Px\|^2 - \|(I - Q)x\|^2 < \|Px\|^2$

so (d) cannot hold true.

(d) $\Leftrightarrow$ (e): Simply observe that $(x, Px) = (x, P^2x) = (Px, Px) = \|Px\|^2$ and analogously $(x, Qx) = \|Qx\|^2$.

Note: You may want to draw a diagram over the implications to convince yourself that all equivalencies have been proven.
Exercise 8.4: First we prove that $P_n \to I$ strongly. Fix any $x \in H$. Since $\bigcup_{n=1}^{\infty} \text{ran}(P_n) = H$, we know that $x \in \text{ran}(P_N)$ for some specific $N$. Then, since $\text{ran}(P_n) \subseteq \text{ran}(P_{n+1})$, we see that $x \in \text{ran}(P_m)$ for any $m \geq N$. Consequently, $P_m x = x$ for any $m \geq N$ so $P_n x \to x$ (very rapidly!).

Next suppose that $||I - P_n|| \to 0$. Then there is some $N$ such that $||I - P_N|| \leq 1/2$. Now observe that $I - P_N$ is itself an orthogonal projection (onto $\text{ker}(P_N)$) so it can only have norms 0 and 1. It follows that $||I - P_N|| = 0$, which is to say that $P_N = I$. Since $H = \text{ran}(P_N) \subseteq \text{ran}(P_{N+1}) \subseteq \text{ran}(P_{N+2}) \subseteq \cdots$ we see that $P_n = I$ for any $n \geq N$.

Problem 1: Let $T(t)$ denote the semigroup defined in Section 7.3 of the textbook. Prove that $T(t) \to I$ strongly as $t \searrow 0$. Prove that $T(t)$ does not converge in norm.

Solution: We consider a slightly more general problem. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis for a Hilbert space $H$, and consider for $t \geq 0$ the operator

$$T(t) f = \sum_{n=1}^{\infty} f_n e^{-n^2 t} e_n.$$ 

We will show that as $t \searrow 0$, $T(t) \to I$ strongly but not in norm.

To show $T(t) \to I$ strongly, fix $f \in H$. Fix $\varepsilon > 0$. Set $f_n = (e_n, f)$ and pick $N$ such that $\sum_{n=N+1}^{\infty} |f_n|^2 < \varepsilon^2$. Then by Parseval

$$||T(t) f - f||^2 = \sum_{n=1}^{N} |f_n (e^{-n^2 t} - 1)|^2 + \sum_{n=N+1}^{\infty} |f_n (e^{-n^2 t} - 1)|^2$$

$$\leq \sum_{n=1}^{N} |f_n (e^{-n^2 t} - 1)|^2 + \sum_{n=N+1}^{\infty} 4 |f_n|^2 \leq \sum_{n=1}^{N} |f_n (e^{-n^2 t} - 1)|^2 + 4 \varepsilon^2.$$ 

Since only finitely many terms depend on $t$, we can now easily take the limit as $t \searrow 0$,

$$\lim_{t \searrow 0} ||T(t) f - f||^2 \leq 4 \varepsilon^2.$$ 

Since $\varepsilon$ was arbitrary, we see that $\lim_{t \searrow 0} ||T(t) f - f|| = 0$.

To show that $T(t)$ does not converge to $I$ in norm, we simply observe that for any $t > 0$

$$||T(t) - I|| \geq \sup_n ||(T(t) - I) e_n|| = \sup_n |e^{-n^2 t} - 1| = 1.$$
**Problem 2:** Suppose $P$ is a projection on a Hilbert space $H$. TFAE:

(a) $P$ is orthogonal, i.e. $\text{ker}(P) = \text{ran}(P)^\perp$.
(b) $P$ is self-adjoint, i.e. $\langle Px, y \rangle = \langle x, Py \rangle \; \forall x, y$.
(c) $\|P\| = 0$ or 1.

**Proof:**

(a) $\Rightarrow$ (b): Assume $\text{ker}(P) = \text{ran}(P)^\perp$. Pick any $x, y \in H$. Then

$$\langle Px, y \rangle = \langle \sum_{x \in \text{ran}(P)} Px, y \rangle = \langle Px + (I - P)y, y \rangle = \langle Px, Py \rangle = \langle x, Py \rangle = \langle x, y \rangle.$$ 

(b) $\Rightarrow$ (c): Assume that (b) holds. Then for any $x$,

$$\|Px\|^2 = \langle Px, Px \rangle = \langle P^2x, x \rangle = \langle Px, x \rangle \leq ||P|| \|x\|,$$

so $\|P\| \leq 1$. Obviously it is possible for $\|P\|$ to be zero. We need to prove that the only possible non-zero value of $\|P\|$ is one. To this end, note that if $P \neq 0$, then $\text{ran}(P) \neq \{0\}$. Now observe that if $x$ is a non-zero element in $\text{ran}(P)$, we have $Px = x$ so $\|P\| \geq 1$.

(c) $\Rightarrow$ (a): Assume that (a) does not hold. Then there exist $x \in \text{ran}(P)$ and $y \in \text{ker}(P)$ such that $(x, y) \neq 0$. Set $\alpha = \frac{(x, y)}{|(x, y)|}$ and $z = \alpha y$. Then $z \in \text{ker}(P)$ and $(x, z) = |(x, y)| \in \mathbb{R}_+$. Set $w = x - z t$.

Then $\|Pw\| = \|x\|$, and

$$\|w\|^2 = \|x\|^2 - 2 t \langle x, z \rangle + t^2 \|z\|^2.$$ 

For small $t$, we see that $\|w\| < \|x\| = \|Pw\|$ so $\|P\| > 1$.

No solution is given for Problem 3 since the problem itself outlines precisely how to solve it — just fill in the details.