Homework set 10 — APPM5450, Spring 2008

From the textbook: 11.18, 11.13, 11.16.

In 11.16, you’re free to assume that $f$ is smooth (or that $f \in S(\mathbb{R}^3)$), if you like. You may also assume that $f \in L^1$ in 11.18, but please return to the problem once we’ve described the action of $\mathcal{F}$ on $L^2$.

**Problem 1:** Let $R$ denote a real number such that $0 < R < \infty$ and define
\[
   f_n(x) = \begin{cases} 
   n \cos(nx) & \text{for } |x| \leq R, \\
   0, & \text{for } |x| > R.
   \end{cases}
\]
For which numbers $R$, if any, is it the case that $f_n \to 0$ in $S^*$?

**Problem 2:** (Optional review of old material.) Prove that $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$. Prove that $C_0(\mathbb{R}^d)$ is a closed subset of $C_b(\mathbb{R}^d)$.

What follows is a set of review questions for Chapter 11. They are not part of the homework but you may find them useful in preparing for the third midterm and the final:

What does it mean for $\varphi_n \to \varphi$ in $S$?

Make absolutely sure that you understand problems like 11.4, 11.10a.

Prove that if $\varphi_n \to \varphi$ in $S$, then $x\varphi_n(x) \to x\varphi(x)$ and $\partial \varphi_n \to \partial \varphi$ in $S$.

Let $T$ be a linear map from $S$ to $\mathbb{R}$. What does it mean for $T$ to be continuous? Prove that if there exists a finite $C$ and a finite $N$ such that $|T(\varphi)| \leq C \sum_{|\alpha|, n \leq N} ||\varphi||_{\alpha, n}$, then $T$ is continuous.

Let $T \in S^*(\mathbb{R}^d)$, and let $\alpha$ be a multi-index. Define $x^\alpha T$. Prove that what you define is a tempered distribution.

Prove that $n^2 \sin(nx) \to 0$ in $S^*$.

Is the Schwartz space dense in $S^*$?

Prove that $\sup_x |x^\beta \partial^\alpha \varphi(x)| < \infty \forall \alpha, \beta$. Prove that $\sup_x |(1 + |x|^2)^{k/2} \partial^\alpha \varphi(x)| < \infty \forall \alpha, k$.

Assume that $\int |f|^2 < \infty$, set $\langle T, \varphi \rangle = \int f \varphi$. Prove that $T \in S^*$.

Let $H$ be a function such that $H(x) = 1$ if $x \geq 0$, zero otherwise. Prove that $H \in S^*$. Calculate $H'$. Let $H_R$ denote the function that is 1 when $0 \leq x \leq R$ and zero otherwise. Prove that $H_R \to H$ in $S^*$ as $R \to \infty$.

Let $\psi$ be a Schwartz function such that $\int \psi = 0$. Set $\varphi_n(x) = n \psi(nx)$. Does $\varphi_n$ converge in $S$?

Does $\varphi_n$ converge in $S^*$?

Prove that $\text{PV}(1/x)$ is a continuous functional on $S$.

What is the distributional derivative of $\text{PV}(1/x)$?

Define $\hat{T}$ for $T \in S^*$. Prove that what you define is a continuous map on $S$. 