Problem 1: (28p) Four points for each question. No motivation required.

(a) State the axioms for a σ-algebra.
(b) Let $H$ be a Hilbert space, and let $A \in \mathcal{B}(H)$. Which statements are necessarily true:
   (i) If $A^* A = I$, then $||A x|| = ||x||$ for all $x \in H$.
   (ii) If $||A x|| = ||x||$ for all $x \in H$, then $(Ax, Ay) = (x, y)$ for all $x, y \in H$.
   (iii) If $(Ax, Ay) = (x, y)$ for all $x, y \in H$, then $A$ is unitary.
(c) Let $(\varphi_n)_{n=1}^\infty$ be a sequence of Schwartz functions on $\mathbb{R}$ that are all supported in the interval $I = [-1, 1]$. Suppose further that
   $$\lim_{n \to \infty} \left( \sup_{x \in I} |\varphi_n(x) - \varphi(x)| \right) = 0.$$ 
   Which of the following statements are necessarily true:
   (i) $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R})$.
   (ii) $\varphi_n \to \varphi$ in $\mathcal{S}'(\mathbb{R})$.
   (iii) $\varphi_n \to \varphi$ in norm in $L^p(\mathbb{R})$ for all $p \in [1, \infty]$.
(d) Define an operator $A$ on $L^2(\mathbb{R})$ via $[A u](x) = \frac{1}{2}(u(x) + u(-x))$. (To be rigorous, we could define $A$ on $\mathcal{S}(\mathbb{R})$ and then extend it to $L^2(\mathbb{R})$ via a density argument.) Specify $\sigma(A)$.
(e) Let $p \in [1, \infty]$, and define functions $(f_n)_{n=1}^\infty \subset L^p(\mathbb{R})$ via $f_n = \frac{1}{\sqrt{n}} \chi_{[0, n]}$. For which $p \in [1, \infty]$ does $(f_n)_{n=1}^\infty$ converge weakly?
(f) Define $f \in \mathcal{S}'(\mathbb{R})$ via $f(x) = \sin(x)$. What is $\hat{f}$?
(g) Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform. What is the spectrum of $\mathcal{F}$?

Solution:

(a) See text book.
(b) (i) is TRUE since $||Ax||^2 = (Ax, Ax) = (A^* Ax, x) = (Ix, x) = ||x||^2$.
   (ii) is TRUE due to the polarization identity.
   (iii) is FALSE since the condition does not imply that the operator is onto (the right-shift operator on $\ell^2(\mathbb{N})$ provides a counter example).
(c) (i) is FALSE since, for instance, $||\varphi_n - \varphi||_{1,0} = ||\varphi' - \varphi'||_u$ need not converge to zero.
   (ii) is TRUE.
   (iii) is TRUE.
(d) $\sigma(A) = \{0, 1\}$. (Note that $A$ is a projection operator.)
(e) For $p \geq 2$, we have $||f_n||_p = n^{\frac{1}{p} - \frac{1}{2}}$. For $p > 2$, we see that $\lim_{n \to \infty} ||f_n||_p = 0$, while for $p < 2$, we have $\lim_{n \to \infty} ||f_n||_p = \infty$ so $(f_n)$ cannot possibly converge weakly. In the borderline case $p = 2$ we have $||f_n||_2 = 1$, but we can show weak convergence by verifying that $(f_n, g) \to 0$ for all $g$ in a dense subset (such as the compactly supported functions).
(f) $\hat{f} = \frac{\sqrt{2\pi}}{2i} \left( \tau_1 \delta - \tau_{-1} \delta \right)$ (so that $(\hat{f}, \varphi) = \frac{\sqrt{2\pi}}{2i} \left( \varphi(-1) - \varphi(1) \right)$). To see this, observe that $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$, that $\mathcal{F}[e^{ikx} \varphi] = \tau_k \hat{\varphi}$, and that $\mathcal{F}[1] = \sqrt{2\pi} \delta$.
(g) $\sigma(\mathcal{F}) = \sigma_p(\mathcal{F}) = \{1, -1, i, -i\}$. Partial credit is given for the answer that $\sigma(\mathcal{F}) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ which you can deduce from the fact that $\mathcal{F}$ is unitary.
Problem 2: (24p) Set $H = L^2(\mathbb{R})$, and consider for $n = 1, 2, 3, \ldots$ the operator $A_n \in \mathcal{B}(H)$ given by

$$[A_n u](x) = e^{-x^2/2n} u(x).$$

Each operator $A_n$ is self-adjoint, and you may use this fact without proving it. Briefly motivate your answers to all questions below except part (c):

(a) (4p) Is $A_n$ compact?
(b) (4p) Is $A_n$ non-negative? Positive? Coercive?
(c) (6p) Specify $\sigma(A_n)$, $\sigma_p(A_n)$, $\sigma_c(A_n)$, and $\sigma_t(A_n)$.
(d) (6p) Does the sequence $(A_n)_{n=1}^\infty$ converge in $\mathcal{B}(H)$? If so, specify the limit and the mode of convergence.
(e) (4p) With $\mathcal{F}$ the Fourier transform, describe the operator $\hat{A}_n = \mathcal{F}^* A_n \mathcal{F} \in \mathcal{B}(H)$.

That is, specify the action of $\hat{A}_n$ without referring to $\mathcal{F}$. Does $(\hat{A}_n)_{n=1}^\infty$ converge?

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Solution:

(a) No, $A_n$ is not compact. To prove this, set $\varphi_j = 2^{j/2} \chi_{(2^{-j-1}, 2^{-j+1})}$. Then $(\varphi_j)_{j=1}^\infty$ is a bounded sequence, but $(A_n \varphi_j)_{j=1}^\infty$ cannot have a convergent subsequence since it is an orthogonal sequence in which the vectors satisfy $\|A_n \varphi_j\| \geq e^{-1/2}$.

(b) $A_n$ is positive (and hence non-negative). To see this, fix a non-zero vector $u$. Then pick an $R$ such that $\int_{|x| \leq R} |u(x)|^2 \, dx = \epsilon > 0$. Then

$$(A_n u, u) = \int_{-\infty}^\infty e^{-x^2/2n} |u(x)|^2 \, dx \geq \int_{-R}^R e^{-x^2/2n} |u(x)|^2 \, dx \geq e^{-R^2/2n} \epsilon > 0.$$

To see that $A_n$ is not coercive, set $\psi_j = \chi_{(j, j+1)}$. Then $\|\psi_j\| = 1$, and $\lim_{j \to \infty} \|A_n \psi_j\| = 0$.

(c) $\sigma(A_n) = \sigma_c(A_n) = [0, 1]$. $\sigma_p(A_n) = \sigma_t(A_n) = \emptyset$.

(d) $(A_n)$ converges strongly to the identity. To prove this, fix any $u \in H$. Then

$$\|A_n u - u\|^2 = \int_{-\infty}^\infty \left(e^{-x^2/2n} - 1\right)^2 |u(x)|^2 \, dx.$$

The integrand in (1) converges pointwise to zero as $n \to \infty$. Moreover, the integrand is dominated by $|u(x)|^2$, and $\int_R |u|^2 < \infty$. Therefore, the LDCT applies, and $\lim_{n \to \infty} \|A_n u - u\|^2 = 0$.

To see that $(A_n)$ cannot converge in norm, set $\psi_j = \chi_{(j, j+1)}$. Then $\|\psi_j\| = 1$, and so $\|A_n - I\| \geq \|(A_n - I) \psi_j\| \geq 1 - e^{-j^2/2n}$. Taking the limit as $j \to \infty$, we see $\|A_n - I\| \geq 1$.

(e) The key observation is that multiplication by a function in physical space corresponds to convolution in Fourier space. To formalize, set $\varphi_n(x) = e^{-x^2/2n}$, and pick $v \in H$. Then

$$\hat{A}_n v = \mathcal{F}^* [A_n [\mathcal{F} v]] = \mathcal{F}^* [A_n \hat{v}] = \mathcal{F}^* [\varphi_n \hat{v}] = \sqrt{2\pi} \varphi_n * \hat{v}.$$

Since $\varphi_n(t) = \sqrt{n} e^{-nt^2/2}$, we find

$$[\hat{A}_n v](t) = \sqrt{n} \sqrt{2\pi} \int_{-\infty}^\infty e^{-n(t-s)^2/2} v(s) \, ds.$$

Finally, observe that since $\mathcal{F}$ is unitary, the convergence properties of $(\hat{A}_n)$ are exactly the same as those of $(A_n)$. In other words, $(\hat{A}_n)$ converges strongly (and not in norm) to $\mathcal{F}^* I \mathcal{F} = I$. 
Problem 3: (18p) Let $p$ be a real number such that $1 \leq p < \infty$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^p(\mathbb{R})$ that converges pointwise to a function $f$. In other words,

$$\lim_{n \to \infty} f_n(x) = f(x), \quad \text{for all } x \in \mathbb{R}.$$ 

Suppose further that all $f_n$ satisfy

$$|f_n(x)| \leq 2|f(x)|, \quad \text{for all } x \in \mathbb{R}.$$ 

For each of the three sets of conditions on $f$ given below, specify for which $r \in [1, \infty)$ it is necessarily the case that

$$\lim_{n \to \infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.$$

(a) $|f| \leq \chi_{[-1,1]}$.

(b) $f \in L^p(\mathbb{R})$ and $|f(x)| \leq 1$ for all $x \in \mathbb{R}$.

(c) $f \in L^p(\mathbb{R})$.

For each part, three points for a correct answer, and three points for a correct motivation.

Solution: 

(a) $r \in [1, \infty)$.  
(b) $r \in [p, \infty)$.  
(c) $r = p$.

To motivate, we need to prove the claim when it is true, and provide counter-examples when it is not. The basic question we need to resolve is when

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} |f(x) - f_n(x)|^r \, dx = 0.$$ 

The integrand in (2) converges to zero pointwise, and we want to bring the LDCT to bear. To this end, we construct a dominator $h$ via

$$|f(x) - f_n(x)|^r \leq (|f(x)| + |f_n(x)|)^r \leq (|f(x)| + 2|f(x)|)^r = 3^r |f(x)|^r =: h(x).$$

We will analyze each of the three assumptions to see when $\int h < \infty$.

(a) If $|f| \leq \chi_{[-1,1]}$, then $h \leq 3^r \chi_{[-1,1]}$ so $\int h < r^3 2 < \infty$ and LDCT applies.

(b) Case 1 - $r \geq p$: In this case, $h(x) = 3^r |f(x)|^r \leq 3^r |f(x)|^p$ since $|f(x)| \leq 1$. Therefore, $\int h \leq \frac{3^r}{r^3} ||f||_p^p < \infty$, and LDCT applies.

Case 2 - $r < p$: In this case, the LDCT does not apply, and we look for a counter-example. Pick a real number $\alpha$ such that $-\frac{1}{r} < \alpha < -\frac{1}{p}$, and set $f(x) = x^\alpha \chi_{[1,\infty)}$. Then $f \in L^p$. Set $f_n = (1 - 1/n) f$. Then $f_n \to f$ pointwise, but $||f - f_n||_r^r = ||(1/n)f||_r^r = \int_1^{\infty} n^{-r} x^{\alpha r} \, dx = \infty$.

(c) Case 1 - $r > p$: When $|f|$ is not necessarily bounded, $|f|^r$ is not bounded by $|f|^p$ and the LDCT does not apply. We look for a counter-example. Pick a real number $\alpha$ such that $-\frac{1}{r} < \alpha < -\frac{1}{r}$, and set $f(x) = x^\alpha \chi_{(0,1)}$. Then $f \in L^p$. Set $f_n = (1 - 1/n) f$. Then $f_n \to f$ pointwise, but $||f - f_n||_r^r = ||(1/n)f||_r^r = \int_0^1 n^{-r} x^{\alpha r} \, dx = \infty$.

Case 2 - $r = p$: In this case, $\int h = \int 3^p |f|^p = 3^p ||f||_p^p < \infty$ so LDCT applies.

Case 3 - $r < p$: In this case, the same counter-example we constructed in part (b) works.

Note: A complete motivation requires counter-examples for the case where the claim does not hold. However, nobody provided them, so only one point was docked for such an omission.
Problem 4: (15p) Let \((c_n)_{n=1}^{\infty}\) be a sequence of complex numbers such that
\[
\sum_{n=1}^{\infty} n^6 |c_n|^2 < \infty,
\]
and set
\[ u(x) = \sum_{n=1}^{\infty} c_n e^{inx}. \]
For which non-negative integers \(k\) is it necessarily the case that \(u \in C^k([-\pi, \pi])\)? Motivate your answer without invoking the Sobolev embedding theorem.

Solution: For \(k = 0, 1, 2\).

Set \(u_N = \sum_{n=1}^{N} c_n e^{inx}\). Then \(u_N \in C^k\) for all \(k\). If we can prove that \((u_N)_{N=1}^{\infty}\) is Cauchy in \(C^k\), then we invoke the fact that \(C^k\) is complete to argue that the limit function \(u \in C^k\).

Set
\[
B = \sum_{n=1}^{\infty} n^6 |c_n|^2 < \infty,
\]
let \(j\) be a non-negative integer, and let \(M\) and \(N\) be integers such that \(M < N\). Then for any \(x\) we find
\[
|\partial^{\bar{j}} (u_N(x) - u_M(x))| = |\partial^{\bar{j}} \sum_{n=M+1}^{N} c_n e^{inx}| = \left| \sum_{n=M+1}^{N} (in)^{\bar{j}} c_n e^{inx} \right| \leq \sum_{n=M+1}^{N} n^\bar{j} |c_n| \leq \{\text{Cauchy-Schwartz}\}
\]
\[
\leq \left( \sum_{n=M+1}^{N} n^{2\bar{j} - 6} \right)^{1/2} \left( \sum_{n=M+1}^{N} n^6 |c_n|^2 \right)^{1/2} \leq \left( \sum_{n=M+1}^{\infty} n^{2\bar{j} - 6} \right)^{1/2} B = D_{M,j} B,
\]
where
\[
D_{M,j} = \left( \sum_{n=M+1}^{\infty} n^{2\bar{j} - 6} \right)^{1/2}.
\]
It follows that
\[
||u_N - u_M||_{C^k} \leq \sum_{j=0}^{k} D_{M,j} B.
\]
Observe that \(\lim_{M \to \infty} D_{M,j} = 0\) when \(2\bar{j} - 6 < -1\). Since \(j\) is an integer, this happens when \(j = 0, 1, 2\).

Note: Most answers to this question consisted of a demonstration that the sum \(\partial^{\bar{k}} u = \sum c_n (in)^{\bar{k}} e^{inx}\) converges in the \(L^2\)-norm when \(\bar{k} \leq 3\). This shows that \(u \in H^3\), not that \(u \in C^3\). To get to \(C^3\), you need to invoke some type of Sobolev embedding results such as the one used above.

Also note that while the question asked for a motivation that did not merely invoke the Sobolev embedding theorem, it can of course be used to arrive at the correct answer. The theorem says that \(H^m(\mathbb{T}^d) \subset C^k(\mathbb{T}^d)\) when \(k < m - d/2\). In our case, we find that \(u \in H^3(\mathbb{T}^1)\), so \(m = 3\) and \(d = 1\). We must have \(k < 3 - 1/2\), or, in other words, \(k = 0, 1, 2\).
**Problem 5:** (15p) Define \( f \in S^*(\mathbb{R}) \) via \( f(x) = \frac{|x| - 1}{1 + |x|} \). Calculate the distributional derivatives \( f' \) and \( f'' \). Please motivate carefully.

**Solution:** Observe that \( f(x) = \frac{1 + |x| - 1}{1 + |x|} = 1 - \frac{1}{1 + |x|} \).

First we evaluate \( f' \). Fix \( \varphi \in S \). Then

\[
\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-\infty}^{\infty} \varphi' + \int_{-\infty}^{0} \frac{1}{1 - x} \varphi' + \int_{0}^{\infty} \frac{1}{1 + x} \varphi' = \varphi(0) - \int_{-\infty}^{0} \varphi - \varphi(0) + \int_{0}^{\infty} \varphi = \langle g, \varphi \rangle
\]

where \( g = f' \) is a regular function given by

\[
f'(x) = g(x) = \frac{\text{sign}(x)}{(1 + |x|)^2}.\]

(The definition of \( g(0) \) is arbitrary.)

Observe that in the calculation above we used that \( \lim_{x \to \pm\infty} \varphi(x) = 0 \) for any \( \varphi \in S \).

Proceeding to \( f'' = g' \), we find

\[
\langle f'', \varphi \rangle = -\langle g', \varphi \rangle = -\int_{-\infty}^{0} \frac{2}{(1 - x)^2} \varphi' - \int_{0}^{\infty} \frac{2}{(1 + x)^3} \varphi' = \varphi(0) - \int_{-\infty}^{0} \frac{2}{(1 - x)^3} \varphi + \varphi(0) - \int_{0}^{\infty} \frac{2}{(1 + x)^3} \varphi.
\]

We see that

\[
f'' = g' = 2\delta + h,
\]

where \( h \) is a regular function given by

\[
h(x) = -\frac{2}{(1 + |x|)^3}.
\]

**Note:** Many solutions given involved sign errors, mistaken calculations of the derivative, etc. Such errors of course only result in a very minor loss of points, but notice that they are entirely unnecessary. The signs are obvious if you simply sketch the graphs of \( f \) and \( f' \). Moreover, away from the origin, \( f \) is a regular function and its distributional derivatives must coincide with its classical derivatives, which can easily be evaluated.