Problem 1: (28p) Four points for each question. No motivation required.

(a) State the axioms for a \( \sigma \)-algebra.

(b) Let \( H \) be a Hilbert space, and let \( A \in \mathcal{B}(H) \). Which statements are necessarily true:
   (i) If \( A^* A = I \), then \( ||A x|| = ||x|| \) for all \( x \in H \).
   (ii) If \( ||A x|| = ||x|| \) for all \( x \in H \), then \( (Ax, Ay) = (x, y) \) for all \( x, y \in H \).
   (iii) If \( (Ax, Ay) = (x, y) \) for all \( x, y \in H \), then \( A \) is unitary.

(c) Let \( (\varphi_n)_{n=1}^{\infty} \) be a sequence of Schwartz functions on \( \mathbb{R} \) that are all supported in the interval \( I = [-1, 1] \). Suppose further that
   \[ \lim_{n \to \infty} \left( \sup_{x \in I} |\varphi_n(x) - \varphi(x)| \right) = 0. \]
   Which of the following statements are necessarily true:
   (i) \( \varphi_n \to \varphi \) in \( \mathcal{S}(\mathbb{R}) \).
   (ii) \( \varphi_n \to \varphi \) in \( \mathcal{S}^*(\mathbb{R}) \).
   (iii) \( \varphi_n \to \varphi \) in norm in \( L^p(\mathbb{R}) \) for all \( p \in [1, \infty] \).

(d) Define an operator \( A \) on \( L^2(\mathbb{R}) \) via \( [Au](x) = \frac{1}{2}(u(x) + u(-x)) \). (To be rigorous, we could define \( A \) on \( \mathcal{S}(\mathbb{R}) \) and then extend it to \( L^2(\mathbb{R}) \) via a density argument.) Specify \( \sigma(A) \).

(e) Let \( p \in [1, \infty] \), and define functions \( (f_n)_{n=1}^{\infty} \subset L^p(\mathbb{R}) \) via \( f_n = \frac{1}{\sqrt{n}} \chi_{[0,n]} \). For which \( p \in [1, \infty] \) does \( (f_n)_{n=1}^{\infty} \) converge weakly?

(f) Define \( f \in \mathcal{S}^*(\mathbb{R}) \) via \( f(x) = \sin(x) \). What is \( \hat{f} \)?

(g) Let \( \mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) denote the Fourier transform. What do you know about the spectrum of \( \mathcal{F} \)?

Problem 2: (24p) Set \( H = L^2(\mathbb{R}) \), and consider for \( n = 1, 2, 3, \ldots \) the operator \( A_n \in \mathcal{B}(H) \) given by
   \[ [A_n u](x) = e^{-x^2/2n} u(x). \]
   Each operator \( A_n \) is self-adjoint, and you may use this fact without proving it. Briefly motivate your answers to all questions below except part (e):

(a) (4p) Is \( A_n \) compact?

(b) (4p) Is \( A_n \) non-negative? Positive? Coercive?

(c) (6p) Specify \( \sigma(A_n) \), \( \sigma_p(A_n) \), \( \sigma_c(A_n) \), and \( \sigma_t(A_n) \).

(d) (6p) Does the sequence \( (A_n)_{n=1}^{\infty} \) converge in \( \mathcal{B}(H) \)? If so, specify the limit and the mode of convergence.

(e) (4p) With \( \mathcal{F} \) the Fourier transform, describe the operator \( \hat{A}_n = \mathcal{F}^* A_n \mathcal{F} \in \mathcal{B}(H) \).
   That is, specify the action of \( \hat{A}_n \) without referring to \( \mathcal{F} \). Does \( (\hat{A}_n)_{n=1}^{\infty} \) converge?
Problem 3: (18p) Let $p$ be a real number such that $1 \leq p < \infty$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^p(\mathbb{R})$ that converges pointwise to a function $f$. In other words,

$$
\lim_{n \to \infty} f_n(x) = f(x), \quad \text{for all } x \in \mathbb{R}.
$$

Suppose further that all $f_n$ satisfy

$$
|f_n(x)| \leq 2|f(x)|, \quad \text{for all } x \in \mathbb{R}.
$$

For each of the three sets of conditions on $f$ given below, specify for which $r \in [1, \infty)$ it is necessarily the case that

$$
\lim_{n \to \infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.
$$

(a) $|f| \leq \chi_{[-1,1]}$.

(b) $f \in L^p(\mathbb{R})$ and $|f(x)| \leq 1$ for all $x \in \mathbb{R}$.

(c) $f \in L^p(\mathbb{R})$.

For each part, three points for a correct answer, and three points for a correct motivation.

Problem 4: (15p) Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers such that

$$
\sum_{n=1}^{\infty} n^6 |c_n|^2 < \infty,
$$

and set

$$
u(x) = \sum_{n=1}^{\infty} c_n e^{i n x}.
$$

For which non-negative integers $k$ is it necessarily the case that $u \in C^k([\pi, \pi])$? Motivate your answer without invoking the Sobolev embedding theorem.

Problem 5: (15p) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = |x|/(1 + |x|)$. Calculate the distributional derivatives $f'$ and $f''$. Please motivate carefully.