Problem 1: Mark the following as TRUE/FALSE. Motivate your answers briefly.

(a) [2p] If \( f_n \to f \) in \( L^2(\mathbb{R}^d) \), then \( \hat{f}_n \to \hat{f} \) in \( L^2(\mathbb{R}^d) \). (Note the weak convergence arrows.)

(b) [2p] Set \( B = \{ f \in L^2(\mathbb{R}^d) : ||f||_2 \leq 1 \} \). Then \( \mathcal{F} \) is a bijection from \( B \) to \( B \).

(c) [2p] Let \( f \) be a function on \( \mathbb{R} \) such that \( \int_{-\infty}^{\infty} (1 + |x|)|f(x)| \, dx < \infty \). Then \( \hat{f} \in C^1(\mathbb{R}) \).

(d) [2p] If \( f_n \to f \) in \( L^1(\mathbb{R}^d) \), then \( \hat{f}_n \to \hat{f} \) uniformly.

(e) [2p] If \( \varphi_n \to \varphi \) in \( S(\mathbb{R}^d) \) and \( \alpha \) is a multi-index, then \( \partial^\alpha \varphi_n \to \partial^\alpha \varphi \) in \( S(\mathbb{R}^d) \).

Problem 2: [7p] Let \( d \) be a positive integer. Prove that if \( s \) is a real number that is “large enough”, then \( H^s(\mathbb{R}^d) \subset C_0(\mathbb{R}^d) \). Make sure to specify what “large enough” is.

Problem 3: Calculate the Fourier transform of the following functions on \( \mathbb{R} \):

(a) [3p] The Dirac \( \delta \)-function.

(b) [3p] \( f(x) = x^k \).

(c) [3p] \( g(x) = \sin(x) \).

Problem 4:

(a) [2p] State the definition of a \( \sigma \)-algebra.

(b) [2p] Is every topology is a \( \sigma \)-algebra? Motivate your answer.

(c*) [2p] Is every \( \sigma \)-algebra a topology? Motivate your answer.

(d) [2p] State the definition of a measure.

(e) [4p] Let (\( X, \mathcal{A}, \mu \)) be a measure space, and let \( \{ \Omega_\beta \}_{\beta \in B} \) be a countable collection of sets in \( \mathcal{A} \). Prove directly from the definition of a measure that

\[
\mu \left( \bigcup_{\beta \in B} \Omega_\beta \right) = \sup \left\{ \mu \left( \bigcup_{\beta \in C} \Omega_\beta \right) : C \text{ is a finite subset of } B \right\}.
\]

Hint: Since \( B \) is countable, you may assume that \( B = \{ 1, 2, 3, \ldots \} \). Then the statement you are asked to prove is equivalent to the statement \( \mu \left( \bigcup_{n=1}^{\infty} \Omega_n \right) = \sup \left\{ \mu \left( \bigcup_{n=1}^{N} \Omega_n \right) : N = 1, 2, 3, \ldots \right\} \).

(f*) [2p] Demonstrate that the formula (*) is not necessarily true if \( B \) is uncountable.

Problem 5: [6p] We define for \( n = 1, 2, 3, \ldots \) functions \( f_n \) on \( \mathbb{R} \) by \( f_n(x) = n^{3/2} \, x e^{-nx^2} \). Either prove that \( (f_n)_{n=1}^{\infty} \) does not converges in \( \mathcal{S}^*(\mathbb{R}) \), or give the limit point and prove convergence.