Problem 1: Let $H$ be a Hilbert space with an ON-basis $(\varphi_j)_{j=1}^{\infty}$, and let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, $(u_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, and $(w_n)_{n=1}^{\infty}$ be sequences in $H$ for which you know the following:

$\langle x_n, x_m \rangle = 0$ if $m \neq n$ and $\langle x_n, x_n \rangle = 1$.

$||y_n|| = 1$

$\limsup_{n \to \infty} ||z_n|| = \infty$

$||u_n|| = 1/n$ and $\lim_{n \to \infty} \langle \varphi_j, u_n \rangle = 0$ for every $j$.

$\lim_{n \to \infty} \langle \varphi_j, v_n \rangle = 0$ for every $j$.

There exists a $w \in H$ such that $||w_n|| \to ||w||$ and $\lim_{n \to \infty} \langle \varphi_j, w_n \rangle = \langle \varphi_j, w \rangle$ for every $j$.

What can you tell about the convergence properties of these six sequences?

In each of the boxes below, enter a “1”, a “2”, or a “3”, as appropriate.

No motivation is required!

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<tr>
<th></th>
<th>$x_n$</th>
<th>$y_n$</th>
<th>$z_n$</th>
<th>$u_n$</th>
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<td>(2) Does not converge strongly.</td>
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<td>(3) May or may not converge strongly.</td>
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(Note that a “3” indicates that not enough information is provided to determine the convergence property that is asked about.)

2 points for each column that has 4 correct answers and 1 point for each column that has 3 correct answers.
Problem 2: Set \( I = [-\pi/2, \pi] \) and consider the Hilbert space \( H = L^2(I) \).

(a) Set \( \varphi_n(x) = \sin(nx) \) and prove that the set \( P = \text{span}(\varphi_n)_{n=1}^\infty \) is not dense in \( H \). (3p)

(b) Set \( e_n(x) = e^{inx}/\sqrt{2\pi} \) and prove that the set \( (e_n)_{n=-\infty}^\infty \) is linearly dependent in the sense that there exists a sequence of complex numbers \( (\alpha_n)_{n=-\infty}^\infty \) such that

\[
0 < \sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty \quad \text{and} \quad \lim_{N \to \infty} \| \sum_{n=-N}^{N} \alpha_n e_n \|_{L^2(I)} = 0.
\]

(4p)

(c) Provide an ON-basis for \( H \). (3p)

Problem 3: Let \( (\lambda_n)_{n=-\infty}^\infty \) denote a bounded sequence of complex numbers and consider the map

(1) \( A : L^2(T) \to l^2(Z) : u \mapsto (\ldots, v_{-1}, v_0, v_1, \ldots) \) where \( v_n = \lambda_n \langle e_n, u \rangle \).

In (1), \( e_n \) denotes the Fourier basis for \( L^2(T) \), \( e_n(x) = e^{inx}/\sqrt{2\pi} \).

(a) Prove that \( \| A \| = \sup_n |\lambda_n| \). (4p)

(b) Let \( F : L^2(T) \to l^2(Z) \) denote the Fourier transform. Complete the following sentences:

\( F^{-1}A \) is self-adjoint if and only if every number \( \lambda_n \) satisfies . . .

\( F^{-1}A \) is unitary if and only if every number \( \lambda_n \) satisfies . . .

Motivate briefly. (6p)

Problem 4: Recall that for an \( n \times n \) matrix \( A \) it is the case that

(2) \( \text{ran}(A) = \ker(A^*)^\perp \).

Now consider the Hilbert space \( H = L^2([-\pi, \pi]) \) and the operator

\( [A u](x) = xe^{ix} u(x) \).

(a) Construct \( A^* \) and prove that (2) does not hold for \( A \). (6p)

(b) Determine \( \| A \| \). (4p)

Problem 5: Let \( H_1 \) and \( H_2 \) be Hilbert spaces.

(a) Define what it means for a map \( U \in B(H_1, H_2) \) to be unitary. (2p)

(b) Suppose that \( A \in B(H_1, H_2) \), that \( A \) is onto, and that \( \| Au \| = \| u \| \) for all \( u \in H_1 \). Is \( A \) necessarily unitary? Motivate briefly. (2p)