Problem 1: Let $H_1$ and $H_2$ be Hilbert spaces, let $U: H_1 \to H_2$ be unitary, and let $A \in \mathcal{B}(H_1)$. Define $	ilde{A} \in \mathcal{B}(H_2)$ by $	ilde{A} = U^* A U$. Prove that

- $\sigma_p(A) = \sigma_p(\tilde{A})$
- $\sigma_c(A) = \sigma_c(\tilde{A})$
- $\sigma_r(A) = \sigma_r(\tilde{A})$

Problem 2: Let $A$ be a self-adjoint compact operator. For $\lambda \in \rho(A)$, set $R_\lambda = (A - \lambda I)^{-1}$ as usual. Construct the spectral decomposition of $R_\lambda$. Use it to prove that $||R_\lambda|| = \frac{1}{\text{dist}(\lambda, \sigma(A))} = \frac{1}{\inf_{\mu \in \sigma(A)} |\lambda - \mu|}$.

Problem 3: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi, \pi]$. Define $\Omega_t = \{u \in H : u(x) = 0 \ \forall \ x \geq t\}$. Note that $\Omega_t$ is a closed linear subspace of $H$. Define $P(t)$ as the orthogonal projection onto $\Omega_t$. Consider the self-adjoint operator $A \in \mathcal{B}(H)$ defined by $[Au](x) = x u(x)$.

(a) Prove that $\Omega_t$ is an invariant subspace of $A$ for every $t \in \mathbb{R}$.

(b) Prove that if $a < b < c < d$, then $\text{ran}(P(b) - P(a)) \perp \text{ran}(P(d) - P(c))$. Conclude that for any numbers $-\pi = t_0 < t_1 < t_2 < \cdots < t_n = \pi$, it is the case that $H = \text{ran}[P(t_1) - P(t_0)] \oplus \text{ran}[P(t_2) - P(t_1)] \oplus \cdots \oplus \text{ran}[P(t_n) - P(t_{n-1})]$, where each term is an invariant subspace of $A$.

(c) For a positive integer $n$, set $h = 2\pi/n$, and $\lambda_j = -\pi + h j$. Define the operator

$$A_n = \sum_{j=1}^{n} \lambda_j (P(\lambda) - P(\lambda_{j-1})).$$

Prove that $||A - A_n|| \leq 2\pi/n$. Conclude that $A_n \to A$ in norm.

Note: There is a spectral theorem for all self-adjoint operators on Hilbert spaces (all normal ones, even). For operators with continuum spectra such as $A$, the spectral decomposition of $A$ involves so called “projection valued measures”. The sum (1) is a Riemann-Riemann-Stieltjes sum of this integral

$$A = \int_{-\pi}^{\pi} \lambda dP(\lambda).$$