Homework set 3 — APPM5450, Spring 2008

From the textbook: 8.6. Optional: 8.5.

Problem 1: Let \(H\) be a Hilbert space, and let \((\varphi_n)_{n=1}^\infty\) denote an orthonormal basis for \(H\). Given a bounded sequence of complex number \((\lambda_n)_{n=1}^\infty\), define the operator \(A\) by setting \(A u = \sum_{n=1}^\infty \lambda_n \varphi_n \langle \varphi_n, u \rangle\).

(a) Prove that \(\|A\| = \sup_n |\lambda_n|\).

(b) Prove that \(A^* u = \sum_{n=1}^\infty \bar{\lambda}_n \varphi_n \langle \varphi_n, u \rangle\). Conclude that \(A\) is self-adjoint iff all \(\lambda_n\)’s are real. When is \(A\) skew-symmetric?

Problem 2: Consider the Hilbert space \(H = L^2([-\pi, \pi])\), and the operator \(A \in B(H)\) defined by \([Au](x) = |x| u(x)\). Prove that \(A\) is self-adjoint and positive, but not coercive. Prove that \(\langle u, v \rangle_A = \langle Au, v \rangle\) is an inner product on \(H\), but that the topology generated by (the norm generated by) this inner product is not equivalent to the topology generated by the \(L^2\)-norm.

Problem 3: Set \(H = l^2(\mathbb{Z})\) and let \(R\) denote the right-shift operator (so that if \(y = Rx\), then \(y_n = x_{n-1}\)). Construct \(R^*\). Prove that \(R\) is unitary. (Recall that the right-shift operator on \(l^2(\mathbb{N})\) is not unitary!)

Problem 4: Consider the Hilbert space \(L^2(\mathbb{T})\). Let \(k\) denote a continuous function on \(\mathbb{T}\) that takes on complex values. Let \(A\) denote the operator \([Au](x) = \int_\mathbb{T} k(x, y) u(y) dy\). Prove that \([A^* u](x) = \int_\mathbb{T} \overline{k(y, x)} u(y) dy\). Conclude that \(A\) is self-adjoint iff \(k(x, y) = \overline{k(y, x)} \forall x, y \in \mathbb{T}\).

Problem 5 (optional): Assume that you have proved that in a Hilbert space, \(\text{ran}(I - K)\) is closed whenever \(K\) has finite rank. Prove from this that \(\text{ran}(I - K)\) is closed for all compact operators \(K\). (Then if you feel industrious, feel free to try proving the statement given in the first sentence.)