Problem 12.8: We want to prove that

\[ ||f - f_n||_p = \int |f - f_n|^p \to \infty. \]

We know that \( |f - f_n|^p \to 0 \) pointwise, so if we can only justify moving the limit inside the integral, we’ll be done.

First note that

\[ |f(x)| = \lim_{n \to \infty} |f_n(x)| \leq |g(x)|. \]

Then we can dominate the integrand as follows:

\[ |f - f_n|^p \leq (|f| + |f_n|)^p \leq (|g| + |g|)^p \leq 2^p|g|^p. \]

Since \( \int |g|^p < \infty \), we find that the Lebesque dominated convergence theorem applies, and so

\[ \lim_{n \to \infty} ||f - f_n||_p^p = \lim_{n \to \infty} \int |f - f_n|^p = \{\text{LDCT}\} = \int (\lim_{n \to \infty} |f - f_n|^p) = \int 0 = 0. \]

Problem 12.16: Fix \( f \in L^p \) and \( \varepsilon > 0 \). We want to prove that there exists a \( \delta > 0 \) such that for \( |h| < \delta \), we have \( ||f - \tau_h f||_p < \varepsilon \).

First pick \( \varphi \in C_c \) such that \( ||f - \varphi||_p < \varepsilon/3 \). Then

\[ ||f - \tau_h f||_p \leq ||f - \varphi||_p + ||\varphi - \tau_h \varphi||_p + ||\tau_h \varphi - \tau_h f||_p \]
\[ = ||f - \varphi||_p + ||\varphi - \tau_h \varphi||_p + ||\varphi - f||_p < \varepsilon/3 + ||\varphi - \tau_h \varphi||_p + \varepsilon/3. \]

Set \( R = \sup\{ |x| : \varphi(x) \neq 0 \} \). Since \( \varphi \) is uniformly continuous, there exists a \( \delta \) such that if \( |x - y| < \delta \), then \( |\varphi(x) - \varphi(y)| < \varepsilon/(3\mu(B_{R+1}(0))^{1/p}) \). Then, if \( h < \min(\delta, 1) \),

\[ ||\varphi - \tau_h \varphi||_p^p = \int_{B_{R+1}(0)} |\varphi(x) - \varphi(x - h)|^p dx < \int_{B_{R+1}(0)} \frac{\varepsilon^p}{3^p \mu(B_{R+1}(0))} dx < \frac{\varepsilon^p}{3^p}. \]

Problem 12.17: Set \( f_n = \chi_{(n-1,n)} \). Then if \( m \neq n \),

\[ ||f_n - f_m||_{\infty} = 1, \]

and for finite \( p \),

\[ ||f_n - f_m||_p = \cdots = 2^{1/p}. \]

It follows that no subsequence of \( (f_n)_{n=1}^{\infty} \) can be Cauchy, and can therefore not converge.

Problem 12.18: Set \( f_n = \chi_{(n-1,n)} \). Let \( (f_{n_j})_{j=1}^{\infty} \) be a subsequence of \( (f_n)_{n=1}^{\infty} \). Define \( g \in L^\infty \) by

\[ g = \sum_{j=1}^{\infty} (-1)^j \chi_{(n_j-1,n_j)}, \]

and define \( \varphi \in (L^1)^* \) via \( \varphi(f) = \int fg \). Then \( \varphi(f_{n_j}) = (-1)^j \) (verify!) and so \( (f_{n_j}) \) cannot converge weakly. Since \( L^1 \) is not reflexive, this does not contradict that Banach-Alaoglu theorem.
Problem 12.13: Set $I = [0, 1]$ and let $\Omega$ be a dense set in $L^\infty(I)$. For $r \in I$, set $f_r = \chi_{[0,r]}$, and pick $x_r \in \Omega \cap B_{1/3}(f_r)$. Since $\|f_r - f_s\| = 1$ if $s \neq r$, we find that $\|x_r - x_s\| \geq \|f_r - f_s\| - \|f_r - x_r\| - \|f_s - x_s\| \geq 1/3$, so all the $x_r$'s are distinct. Therefore, $\Omega$ must be uncountable, and $L^\infty$ cannot be separable.

To prove that $C(I)$ cannot be dense in $L^\infty(I)$, simply note that if $f = \chi_{[0,1/2]}$, and $\varphi \in C(I)$, then

$$\|f - \varphi\|_\infty \geq \max(\|\varphi(1/2)\|, |1 - \varphi(1/2)|) \geq 1/2$$

(verify this!).

An alternative argument for why $C(I)$ cannot be dense in $L^\infty(I)$: If $\varphi_n \in C(I)$, and $\varphi_n \to f$ in the supnorm, then $(\varphi_n)$ is a Cauchy sequence with respect to the uniform norm (when applied to continuous functions, the uniform norm and the $L^\infty$ norms are identical). Therefore, there exists a continuous function $\varphi$ such that $\varphi_n \to \varphi$ uniformly. Then $f(x) = \varphi(x)$ almost everywhere. But not every equivalence class function in $L^\infty$ has a continuous function in it (for instance $f = \chi_{[0,1/2]}$).

Problem 12.14: Let $p$ and $q$ be such that $1 \leq p < q \leq \infty$.

First we construct a function $f \in L^p \setminus L^q$. Let $\alpha$ be a non-negative number and set $f(x) = x^{-\alpha} \chi_{[0,1]}$. Then

$$\|f\|_p^p = \int_0^1 x^{-\alpha p} \, dx,$$

which is finite if $\alpha p < 1$. Moreover

$$\|f\|_q^q = \int_0^1 x^{-\alpha q} \, dx$$

which is infinite if $\alpha q > 1$. Consequently, $f \in L^p \setminus L^q$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$ 

To construct a function $f \in L^q \setminus L^p$, set $f = x^{-\alpha} \chi_{(1,\infty)}$. Then

$$\|f\|_p^p = \int_1^\infty x^{-\alpha p} \, dx$$

which is infinite if $\alpha p < 1$. Moreover

$$\|f\|_q^q = \int_1^\infty x^{-\alpha q} \, dx$$

which is finite if $\alpha q > 1$. Thus, $f \in L^q \setminus L^p$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$ 

(The arguments above need slight modifications if $q = \infty$, but the idea is the same.)

Consider the function

$$f(x) = \frac{1}{(|x| (1 + \log^2 |x|))^{1/2}}.$$
That $f \in L^2$ is clear, since
\[
\|f\|_2^2 = \int_{-\infty}^{\infty} \frac{1}{|x|(1 + \log^2 |x|)} \, dx = 2 \int_0^{\infty} \frac{1}{x(1 + \log^2 x)} \, dx = \{x = e^t\}
\]
\[
2 \int_{-\infty}^{\infty} \frac{1}{e^t(1 + t^2)} e^t \, dt = 2\pi.
\]
Moreover, if $p > 2$, then note that there exists a $\delta > 0$ such that
\[
x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1
\]
when $x \in (0, \delta)$. Then
\[
\|f\|_p^p \geq \int_0^{\delta} \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} \, dx = \int_0^{\delta} \frac{1}{x^{(p-2)/2} (1 + \log^2 x)^{p/2}} \, dx = \infty.
\]
Analogously, if $p < 2$, then there exists an $M$ such that
\[
x^{(p-2)/2}(1 + \log^2 x)^{p/2} \leq 1
\]
when $x \geq M$. Then
\[
\|f\|_p^p \geq \int_M^{\infty} \frac{1}{x^{p/2}(1 + \log^2 x)^{p/2}} \, dx = \int_M^{\infty} \frac{1}{x^{(p-2)/2} (1 + \log^2 x)^{p/2}} \, dx = \infty.
\]

**Problem 12.15**: Let $\alpha \in (0, 1)$, and let $m, n \in (1, \infty)$ be such that $1/m + 1/n = 1$ (we will determine suitable values for $\alpha, m, n$ later). Then from Hölder’s inequality we obtain
\[
\|f\|_r^r = \int |f|^r = \int |f|^{\alpha r} |f|^{(1-\alpha)r} \leq \left( \int |f|^{\alpha m r} \right)^{1/m} \left( \int |f|^{(1-\alpha) n r} \right)^{1/n}.
\]
In order to obtain the desired right hand side, we must pick $\alpha, m, n$ so that
\[
\alpha m r = p,
\]
\[
(1-\alpha) n r = q,
\]
\[
(1/m) + (1/n) = 1.
\]
To obtain and equation for $\alpha$, we eliminate $m$ and $n$:
\[
\frac{(1-\alpha)r}{q} = \frac{1}{n} = 1 - \frac{1}{m} = 1 - \frac{\alpha r}{p}.
\]
Solving for $\alpha$ we obtain
\[
\alpha = \frac{pq - pr}{rq - rp} = \frac{1/r - 1/q}{1/p - 1/q}.
\]
Equation (1) now takes the form
\[
\|f\|_r \leq \left( \left( \|f\|_p^p \right)^{1/m} \left( \|f\|_q^q \right)^{1/n} \right)^{1/r} = \|f\|_{p^{m/r}} \|f\|_{q^{n/r}}.
\]
Finally note that

\[ \frac{p}{mr} = \alpha = \frac{1}{r} - \frac{1}{q} \quad \frac{1}{r} - \frac{1}{q}, \]

\[ \frac{q}{nr} = 1 - \alpha = 1 - \frac{1}{r} \quad \frac{1}{r} - \frac{1}{q} = \frac{1}{p} - \frac{1}{q}. \]