Homework 13

12.4) Give an example of a monotonic decreasing sequence of nonnegative functions converging pointwise to a function \( f \) such that the equality in Theorem 12.33 (Monotone convergence) does not hold.

Consider \( f_n(x) = \frac{1}{n} \) for all \( x \in \mathbb{R} \). Then \( \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) \, dx = \infty \), whereas \( \int_{-\infty}^{\infty} f_n(x) \, dx = 0 \).

Problem 1) Let \( (f_n)_{n=1}^{\infty} \) be a sequence of real valued measurable functions on \( \mathbb{R} \) such that \( \lim_{n \to \infty} f_n(x) = x \) for all \( x \in \mathbb{R} \). Specify which of the following limits necessarily exist, and give a formula for the limit in the cases where this is possible:

1. \( \lim_{n \to \infty} \int_{1}^{2} \frac{f_n(x)}{1 + f_n(x)^2} \, dx \)

   We can bound the integrand:
   \[
   \left| \frac{f_n(x)}{1 + f_n(x)^2} \right| \leq \sup_t \frac{|t|}{1 + t^2} \leq 1
   \]

   Then, since \( \int_{1}^{2} 1 \, dx = 1 < \infty \) dominated convergence applies:
   \[
   \lim_{n \to \infty} \int_{1}^{2} \frac{f_n(x)}{1 + f_n(x)^2} \, dx = \int_{1}^{2} \lim_{n \to \infty} \frac{f_n(x)}{1 + f_n(x)^2} \, dx = \int_{1}^{2} \frac{x}{1 + x^2} \, dx = \left[ \log(1 + x^2) \right]_1^2 = \log \left( \frac{5}{2} \right)
   \]

2. \( \lim_{n \to \infty} \int_{0}^{\infty} \frac{\sin(f_n(x))}{f_n(x)} \, dx \)

   We can bound the integrand:
   \[
   \left| \frac{\sin(f_n(x))}{f_n(x)} \right| \leq \left| \frac{\sin(t)}{t} \right| \leq 1
   \]

   Then, since \( \int_{0}^{\infty} 1 \, dx = 1 < \infty \) dominated convergence applies:
   \[
   \lim_{n \to \infty} \int_{0}^{\infty} \frac{\sin(f_n(x))}{f_n(x)} \, dx = \int_{0}^{\infty} \lim_{n \to \infty} \frac{\sin(f_n(x))}{f_n(x)} \, dx = \int_{0}^{\infty} \frac{\sin(x)}{x} \, dx \approx 0.946083
   \]
We can bound the integrand: \( \left| \frac{\sin(f_n(x))}{f_n(x)} \right| \leq \frac{|\sin(t)|}{t} \leq 1 \)

However, since \( \int_0^\infty 1 \, dx = \infty \) dominated convergence does not apply.

For this problem we can actually achieve different values for the limit depending on \( f_n(x) \).

a) Define \( f_n(x) = \begin{cases} x & 0 \leq x \leq 2\pi n \\ \pi & x > 2\pi n \end{cases} \), then \( \lim_{n \to \infty} \int_0^\infty \frac{\sin(f_n(x))}{f_n(x)} \, dx = \frac{\pi}{2} \)

b) Note that \( \frac{\sin(f_n(x))}{f_n(x)} \) oscillates about the x-axis with decreasing magnitude. For each \( n \) we can construct \( f_n(x) \) so that \( \frac{\sin(f_n(x))}{f_n(x)} \) is made up of \( 2n \) sections of area above the x-axis while counting just \( n \) sections of area below the x-axis. Then \( \lim_{n \to \infty} \int_0^\infty \frac{\sin(f_n(x))}{f_n(x)} \, dx = \infty \)

Since every term in the sum is non-negative monotonic convergence applies:
\[
\lim_{N \to \infty} \int_0^\infty \sum_{n=1}^N \frac{1}{n^2 \left(1 + \left| f_n(x) \right|^2 \right)} \, dx < \infty
\]
We know that the limit exists and is finite, but what the actual limit is depends on \( (f_n)_{n=1}^\infty \).

Since every term in the sum is non-negative monotonic convergence applies:
\[
\lim_{N \to \infty} \int_0^\infty \sum_{n=1}^N \frac{1}{n^2 \left(1 + \left| f_n(x) \right|^2 \right)} \, dx = \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^2 \left(1 + f_n(x)^2 \right)} \, dx
\]

Once again the limit exists, but now (depending on \( (f_n)_{n=1}^\infty \)) it might be infinite (the key difference is that the interval is no longer finite). Consider:

a) \( f_n(x) = x \) for all \( n \). Then \( \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^2 \left(1 + x^2 \right)} \, dx = \sum_{n=1}^\infty \frac{1}{n^2} \int_0^\infty \frac{1}{1 + x^2} \, dx = \frac{\pi^2}{6} \frac{\pi}{2} = \frac{\pi^3}{12} \)

b) \( f_n(x) = \begin{cases} x & 0 \leq x \leq n \\ 0 & x > n \end{cases} \). Then the integral is infinite.
Problem 2) Let $(f_n)_{n=1}^\infty$ be a sequence of real valued measurable functions on $R$ such that $|f_n(x)| \leq 1$ and $\lim_{n \to \infty} f_n(x) = 1$ for all $x \in R$. Evaluate the following (justify your calculation):

$$\lim_{n \to \infty} \int_R f_n(x)e^{-\frac{1}{2}(x-2\pi)^2} \, dx$$

$$\lim_{n \to \infty} \int_R f_n(x)e^{-\frac{1}{2}(x-2\pi)^2} \, dx = \lim_{n \to \infty} \int_R f_n(x)e^{-\frac{1}{2}(y+2\pi n)^2} \, dy = \lim_{n \to \infty} \int_R f_n(x)e^{-\frac{1}{2}y^2} \, dy = (*)$$

Note that the first equality is a substitution and the second uses the periodicity of cosine.

For all $y$ we have $f_n(y) e^{-\frac{1}{2}y^2} \xrightarrow{n \to \infty} e^{-\frac{1}{2}y^2}$ and $\left| f_n(y) e^{-\frac{1}{2}y^2} \right| \leq e^{-\frac{1}{2}y^2}$

Then, since $\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \, dy < \infty$, dominated convergence applies:

$$(*) = \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(y) e^{-\frac{1}{2}y^2} \, dy = \int_{-\infty}^{\infty} \lim_{n \to \infty} f_n(y) e^{-\frac{1}{2}y^2} \, dy = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \, dy = \sqrt{2\pi}$$
Problem 3) The solution to this problem is mostly provided as a hint on the homework page. Below the holes in the solution (given as questions in the hint) are filled in.

(3) What can you tell about $\Omega^k_{mn}$ in light of (2)?
You can conclude that $\mu\left(\Omega^k_{mn}\right) = 0$

(4) What do you know about $\Omega^k$ in view of your conclusion from (3)?
$\mu\left(\bigcup_{m,n=N_k}^\infty \Omega^k_{mn}\right) \leq \sum_{m,n=N_k}^\infty \mu\left(\Omega^k_{mn}\right) = 0$

(5) What do you know about $\Omega$ in view of your conclusion from (4)?
$\mu(\Omega) \leq \sum_k \mu(\Omega^k) = 0$

(6) What can you tell about $(f_n(x))_{n=1}^\infty$ for $x \in \Omega$?  
Because $(f_n(x))_{n=1}^\infty$ is Cauchy for $x \in \Omega$ it makes sense to define $f(x) = \lim_{n \to \infty} f_n(x)$ in this region.  
For $x \in \Omega$ we can simply set $f(x) = 0$.
Fix $\varepsilon > 0$. Pick $k > 1/\varepsilon$. Then, for $n \geq N_k$ we have:

$$
\|f - f_n\| = \sup_{x \in \bar{\Omega}} |f(x) - f_n(x)| = \sup_{x \in \bar{\Omega}} |f(x) - f_n(x)| = \sup_{x \in \bar{\Omega}} \lim_{m \to \infty} f_m(x) - f_n(x) \\
\leq \limsup_{m \to \infty} f_m(x) - f_n(x) \leq \frac{1}{k} < \varepsilon
$$

Note that the equality denoted by “(5)” uses $\mu(\Omega^c) = 0$ (proved in (5) above).

Because $\varepsilon$ was arbitrary this implies that $\|f - f_n\| \to 0$