Homework set 13 — APPM5450, Spring 2008

From the textbook: 12.4. Problems 1 and 2 below are important, problem 3 less so.

**Problem 1:** Let \((f_n)_{n=1}^\infty\) be a sequence of real valued measurable functions on \(\mathbb{R}\) such that \(\lim_{n \to \infty} f_n(x) = x\) for all \(x \in \mathbb{R}\). Specify which of the following limits necessarily exist, and give a formula for the limit in the cases where this is possible:

1. \(\lim_{n \to \infty} \int_1^2 \frac{f_n(x)}{1 + f_n(x)^2} \, dx\),
2. \(\lim_{n \to \infty} \int_0^1 \frac{\sin(f_n(x))}{f_n(x)} \, dx\),
3. \(\lim_{n \to \infty} \int_0^\infty \frac{\sin(f_n(x))}{f_n(x)} \, dx\),
4. \(\lim_{N \to \infty} \int_0^1 \sum_{n=1}^N \frac{|f_n(x)|}{n^2(1 + |f_n(x)|)} \, dx\),
5. \(\lim_{N \to \infty} \int_0^\infty \sum_{n=1}^N \frac{1}{n^2(1 + |f_n(x)|^2)} \, dx\).

**Problem 2:** Let \((f_n)_{n=1}^\infty\) be a sequence of real-valued measurable functions on \(\mathbb{R}\) such that \(|f_n(x)| \leq 1\) and \(\lim_{n \to \infty} f_n(x) = 1\) for all \(x\). Evaluate

\[ \lim_{n \to \infty} \int_{\mathbb{R}} f_n(\cos x) e^{-\frac{1}{2}(x-2\pi n)^2} \, dx. \]

Make sure to justify your calculation.

**Problem 3:** Let \((X, \mu)\) be a measure space and consider the space \(L^\infty(X, \mu)\) consisting of all measurable functions from \(X\) to \(\mathbb{R}\) such that

\[ ||f||_\infty = \operatorname{ess sup}_{x \in X} |f(x)| < \infty. \]

Prove that \(L^\infty(X, \mu)\) is closed under the norm \(|| \cdot ||_\infty\).

**Hint:** You may want to start as follows:

1. Let \((f_n)_{n=1}^\infty\) be a Cauchy sequence in \(L^\infty(X, \mu)\).
2. For each positive integer \(k\), there exists and \(N_k\) such that for \(m, n \geq N_k\), \(||f_n - f_m||_\infty < 1/k\).
3. For each \(k\), and for each \(m, n \geq N_k\), let \(\Omega_{mn}^k\) denote the set of all \(x \in X\) such that \(|f_m(x) - f_n(x)| < 1/k\). What can you tell about \(\Omega_{mn}^k\) in light of (2)?
4. Form \(\Omega^k = \cap_{m,n=N_k}^\infty \Omega_{mn}^k\). What do you know about \(\Omega^k\) in view of your conclusion from (3)?
5. Form \(\Omega = \cap_{k=1}^\infty \Omega^k\). What do you know about \(\Omega\) in view of your conclusion from (4)?
6. What can you tell about \((f_n(x))_{n=1}^\infty\) for \(x \in \Omega\)?