Weak differentiation on $L^2(\mathbb{T})$ (without Fourier methods)

We consider the space $X = L^2(\mathbb{T})$, with the usual norm

$$||f|| = \left( \int_I |f(x)|^2 \, dx \right)^{1/2}.$$  

Let $\Omega$ denote the set of continuously differentiable functions on $\mathbb{T}$. Note that $\Omega$ is dense$^1$ in $X$.

Fix a function $f \in X$ and define for $g \in \Omega$ the functional

$$T_f(g) = -\int_\mathbb{T} f(x) g'(x) \, dx.$$  

Suppose that there exists a number $C$ (that depends on $f$) such that

$$|T_f(g)| \leq C ||g||, \quad \forall \ g \in \Omega.$$  

Then $T_f$ is a continuous functional defined on a dense set. It follows that $T_f$ has a unique extension $\tilde{T}_f \in X^*$. By the Riesz representation theorem, we know that there exists a unique $h \in X$ such that

$$\tilde{T}_f(g) = \langle h, g \rangle, \quad \forall \ g \in X.$$  

We define this function $h$ to be the weak derivative of $f$.

Remark 1: If $f$ is a classically differentiable function, then our definition of a weak derivative coincides with the classical definition. To see this, note that if $f \in \Omega$, then using integration by parts, we obtain

$$T_f(g) = -\int_\mathbb{T} f(x) g'(x) \, dx = \int_\mathbb{T} f'(x) g(x) \, dx = \langle f', g \rangle.$$  

It follows that in this case

$$\langle f', g \rangle = \langle h, g \rangle \quad \forall \ g \in \Omega,$$

and since $\Omega$ is dense in $X$, we must have $f' = h$.

Remark 2: The definition of a weak derivative given here coincides with the Fourier definition. To see this, note that if $g \in \Omega$, and $f = \sum \alpha_n e_n$ and $g = \sum \beta_n e_n$, then

$$T_f(g) = -\langle f, g' \rangle = -\sum_{n \in \mathbb{Z}} \alpha_n \beta_n n = \sum_{n \in \mathbb{Z}} \beta_n \alpha_n.$$  

Since $\Omega$ is dense in $X$, it follows that the number

$$C = \sup_{g \in \Omega} |T_f(g)|$$

is finite if and only if $(i n \alpha_n)_{n=-\infty}^{\infty} \in L^2(\mathbb{Z})$, and if it is, then necessarily $h = \sum i n \alpha_n e_n$.

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$^1$To see that $\Omega$ is dense in $X$, note that $\Omega$ contains the set $\mathcal{P}$ of all polynomials on $I$, and $\mathcal{P}$ is dense in $X$. 

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