Problem 1: Pick out the true statements from the list below. One point each, no motivation required.

(a) If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}$.
(b) If $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}^*$.
(c) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in C_0(\mathbb{R}^d)$.
(d) If $f \in H^s(\mathbb{R}^d)$ and $s > 1/2$, then $f \in C_0(\mathbb{R}^d)$.
(e) If $f \in C_0(\mathbb{R}^d)$, then $\hat{f} \in L^2(\mathbb{R}^d)$.
(f) If $f, g \in L^2(\mathbb{R}^d)$, then $\langle f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R}^d)}$.

(a) True. (Since $\mathcal{F}$ is continuous on $\mathcal{S}$.)
(b) True. (Since $\mathcal{F}$ is continuous on $\mathcal{S}$, we know that $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}$; and since convergence in $\mathcal{S}$ implies convergence in $\mathcal{S}^*$, it follows that $\hat{\varphi}_n \to \hat{\varphi}$ in $\mathcal{S}^*$ as well.)
(c) True. (The Riemann-Lebesgue lemma states that in fact $\hat{f} \in C_0(\mathbb{R}^d).$)
(d) Not true unless $d = 1$. (In the general case, $s > d/2$ is required.)
(e) Not true. (If it were, then we'd have $f \in L^2$ since $\mathcal{F}^{-1}$ is a unitary map on $L^2$. But not every function in $C_0$ belongs to $L^2$.)
(f) True. ($\mathcal{F}$ is a unitary map on $L^2(\mathbb{R}^d)$.)
Problem 2: Suppose that \((a_n)_{n=1}^{\infty}\) are real numbers such that \(\sum_{n=1}^{\infty} |a_n| < \infty\). Set \(f(x) = \sum_{n=1}^{\infty} a_n e^{inx}\). Is it necessarily the case that \(\int_{-\pi}^{\pi} f(x) \, dx = 0\)? Motivate your answer. (4p)

Yes, \(\int f = 0\). To prove this, set \(f_N(x) = \sum_{n=1}^{N} a_n e^{inx}\). Then

\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \left( \lim_{N \to \infty} f_N(x) \right) \, dx.
\]

Now set

\[
g(x) = \sum_{n=1}^{\infty} |a_n|.
\]

Then \(|f_N(x)| \leq g(x)\) for all \(x\), and \(\int_{-\pi}^{\pi} g(x) \, dx < \infty\). The Lebesgue dominated convergence theorem now allows us to swap the integral and the limit in (1), and so

\[
\int_{-\pi}^{\pi} f(x) \, dx = \lim_{N \to \infty} \int_{-\pi}^{\pi} f_N(x) \, dx.
\]

Finally note that

\[
\int_{-\pi}^{\pi} f_N(x) \, dx = \sum_{n=1}^{N} a_n \int_{-\pi}^{\pi} e^{inx} \, dx = \sum_{n=1}^{N} a_n \int_{-\pi}^{\pi} e^{inx} \, dx = 0,
\]

since \(\int_{-\pi}^{\pi} e^{inx} \, dx = 0\) for any positive integer \(n\).
Problem 3: For \( n = 1, 2, 3, \ldots \), set \( T_n(x) = \sin(n x) \chi_{[-n, n]}(x) \). Does the sequence \( (T_n)_{n=1}^\infty \) converge in \( \mathcal{S}^*(\mathbb{R}) \)? Motivate your answer. (4p)

Fix \( \varphi \in \mathcal{S} \). Then

\[
|\langle T_n, \varphi \rangle| = \left| \int_{-n}^{n} \sin(n x) \varphi(x) \, dx \right|
\]
\[
= \left| \left[ -\frac{\cos(n x)}{n} \varphi(x) \right]_{-n}^{n} + \int_{-n}^{n} \frac{\cos(n x)}{n} \varphi'(x) \, dx \right|
\]
\[
= \left| -\frac{\cos(n^2)}{n} \varphi(n) + \frac{\cos(n^2)}{n} \varphi(-n) + \int_{-n}^{n} \frac{\cos(n x)}{n} \varphi'(x) \, dx \right|
\]
\[
\leq \left| \frac{\varphi(n)}{n} \right| + \left| \frac{\varphi(-n)}{n} \right| + \frac{1}{n} \int_{-n}^{n} |\varphi'(x)| \, dx
\]
\[
\leq \frac{2||\varphi||_{0,0}}{n} + \frac{1}{n} \int_{-\infty}^{\infty} \frac{1}{1 + x^2} (1 + x^2) |\varphi'(x)| \, dx
\]
\[
\leq \frac{2||\varphi||_{0,0}}{n} + \frac{1}{n} \pi ||\varphi||_{1,2}.
\]

Consequently, \( \langle T_n, \varphi \rangle \to 0 \) as \( n \to \infty \), and so \( T_n \to 0 \) in \( \mathcal{S}^* \).
Problem 4: Let \( f \) and \( h \) be functions in \( L^2(\mathbb{R}) \). Suppose that \((f_n)_{n=1}^\infty\) is a sequence of functions in \( L^2(\mathbb{R}) \) that converges pointwise to \( f \). Set
\[
\alpha_n = \int_{\mathbb{R}} f_n(x) h(x) \, dx, \quad \text{and} \quad \alpha = \int_{\mathbb{R}} f(x) h(x) \, dx.
\]
(a) Give examples of functions \( f, h \), and \((f_n)_{n=1}^\infty\) as described above such that the numbers \( \alpha_n \) do not converge to \( \alpha \). (3p)

(b) Suppose that \(|f_n(x)| \leq 1/(1 + |x|)\) for all \( x \). Prove that then \( \alpha_n \to \alpha \). (3p)

(a) One example is \( h(x) = 1/(1 + |x|) \) and \( f_n(x) = n^2 \chi_{[n,n+1]}(x) \). Then \( f_n \to 0 \) pointwise, so \( \alpha = 0 \), but
\[
\alpha_n = \int_n^{n+1} n^2 \frac{1}{1 + x} \, dx \geq \frac{n^2}{n + 1} \to \infty.
\]

(b) Set \( u_n(x) = f_n(x) h(x) \) and \( u(x) = f(x) h(x) \). Then \( u_n \to u \) pointwise. Setting \( g(x) = (1/(1 + |x|)) |h(x)| \), we have \(|u_n(x)| \leq g(x)\) for all \( x \). Moreover, a simple application of the Cauchy-Schwartz inequality yields
\[
\int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} \frac{1}{1 + |x|} |h(x)| \, dx \leq \left[ \int_{\mathbb{R}} \frac{1}{(1 + |x|)^2} \, dx \int_{\mathbb{R}} |h(x)|^2 \, dx \right]^{1/2},
\]
which is finite since both \( h \) and \((1 + |x|)^{-1}\) are members of \( L^2(\mathbb{R}) \).

Now according to the Lebesgue dominated convergence theorem,
\[
\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \int_{\mathbb{R}} u_n(x) \, dx = \int_{\mathbb{R}} \left( \lim_{n \to \infty} u_n(x) \right) \, dx \int_{\mathbb{R}} f(x) h(x) \, dx = \alpha.
\]

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\(^1\)Cauchy-Schwartz is a little bit of overkill. The simple inequality \(|ab| \leq \frac{1}{2} |a|^2 + \frac{1}{2} |b|^2\) suffices:
\[
\int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} \frac{1}{1 + |x|} |h(x)| \, dx \leq \frac{1}{2} \int_{\mathbb{R}} \left( \frac{1}{(1 + |x|)^2} + |h(x)|^2 \right) \, dx < \infty.
\]
This problem has been corrected: The norm that was originally in the problem has been substituted for a metric.

**Problem 5:** Let $X$ be a set and let $d$ be a metric on $X$. We define a collection $S$ of subsets of $X$ by saying that $\Omega \in S$ if and only if for every $x \in \Omega$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \Omega$, where $B_\varepsilon(x) = \{ y \in X : d(x, y) < \varepsilon \}$.

The following questions are 1p each. Motivate your answers to (b) and (c) briefly.

(a) State the axioms that a $\sigma$-algebra must satisfy.

(b) Give an example of an uncountable set $X$ and a metric $d$ such that $S$ is a $\sigma$-algebra.

(c) Give an example of an uncountable set $X$ and a metric $d$ such that $S$ is not a $\sigma$-algebra.

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(a) See the textbook.

(b) Set $X = \mathbb{R}$ and

$$d(x, y) = \begin{cases} 1, & \text{when } x = y, \\ 0, & \text{when } x \neq y. \end{cases}$$

Then $S$ is the power set (if $\Omega$ is an arbitrary subset, and $x \in \Omega$, then $B_{1/2}(x) \subseteq \Omega$), which trivially implies that it satisfies all the axioms of a $\sigma$-algebra.

(c) Set $X = \mathbb{R}$ and $d(x, y) = |x - y|$ (the standard metric on $\mathbb{R}$). Then $S$ is the standard topology on $\mathbb{R}$, which is not a $\sigma$-algebra. To see this, note for instance that $\Omega = (0, \infty) \in S$, but $\Omega^c = (-\infty, 0] \notin S$. 