Problem 1: Which of the following are true (no motivation required): (2p in total)
(a) In a Hilbert space, any bounded sequence has a weakly convergent subsequence.
(b) If \( f, g \in C(T) \), then \( ||f * g||_u \leq ||f||_{L^2} ||g||_{L^2} \).
(c) The functions \( (\sin(nx))_{n=1}^{\infty} \) form an orthogonal basis for \( L^2([0, \pi]) \).

(a) True - follows from the Banach-Alaoglu theorem.
(b) True - follows from Cauchy-Schwartz (\( \langle f * g \rangle(t) = \langle f, g_t \rangle \) where \( g_t(x) = g(t - x) \)).
(c) True - see Exercise 7.3.

Problem 2: Let \( A \) be a self-adjoint operator on a Hilbert space \( H \), and let \( \lambda \) be a complex number. Prove that the adjoint of \( \lambda A \) is \( \bar{\lambda} A \). For which \( \lambda \) is \( \lambda A \) necessarily skew-adjoint? (2p)

For any \( x, y \in H \), we find that
\[
\langle (\lambda A)x, y \rangle = \overline{\lambda} \langle Ax, y \rangle = \overline{\lambda} \langle x, A^* y \rangle = \langle x, (\overline{\lambda} A^*)y \rangle.
\]
Consequently, \( (\lambda A)^* = -\lambda A \iff \lambda = -\lambda \iff \text{Re}(\lambda) = 0 \).

Problem 3: Let \( u \) be a function in \( L^2(T) \) and set \( \alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} u(x) \, dx \), for \( n \in \mathbb{Z} \). Obviously, if only finitely many \( \alpha_n \)'s are non-zero, \( u \) will be continuous. Can you give a more general condition involving only the sequence \( (\alpha_n)_{n=-\infty}^{\infty} \)? (2p)

The Sobolev embedding theorem says that \( u \) is continuous if
\[
\sum_{n=-\infty}^{\infty} |n|^{2k} |\alpha_n|^2 < \infty
\]
for some \( k > 1/2 \).
Problem 4: Let $H$ be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ be an orthonormal basis for $H$. Consider for $t \in \mathbb{R}$ the operator $A(t) \in \mathcal{B}(H)$ defined by

$$A(t) u = \sum_{n=1}^{\infty} \left( \frac{1 + it}{1 - it} \right)^n \langle \varphi_n, u \rangle \varphi_n.$$ 

(a) Prove that for any $t \in \mathbb{R}$, the operator $A(t)$ is unitary. (2p)

(b) Is it the case that $A(t)$ is either self-adjoint of skew-adjoint for any $t$? (2p)

(c) For $p \in \mathbb{N}$, set $A_p = A(1/p)$. Does the sequence $(A_p)_{p=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, specify in which sense, and what the limit is. Motivate your answer. (4p)

Set $\lambda_n(t) = \left( \frac{1 + it}{1 - it} \right)^n$.

It follows immediately from Parseval’s equality that

(1) \[ A(t)^* u = \sum_{n=1}^{\infty} \lambda_n(t) \langle \varphi_n, u \rangle \varphi_n = \sum_{n=1}^{\infty} \lambda_n(-t) \langle \varphi_n, u \rangle \varphi_n = A(-t) u. \]

(a) Since $\lambda_n(t)^{-1} = \lambda_n(-t)$, it follows that $A(t)$ is invertible and that $A(t)^{-1} = A(-t)$. That $A(t)$ is unitary is now obvious since $A(t)^* = A(-t) = A(t)^{-1}$.

(b) We find that $A(t)$ is self-adjoint iff every $\lambda_n(t)$ is a real number. This happens only for $t = 0$. Similarly, $A(t)$ is skew-adjoint iff every $\lambda_n(t)$ is a purely imaginary number. That never happens.

(c) $A_p$ converges strongly to the identity operator, but it does not converge in norm.

We first prove that $A_p \to I$ strongly. Fix $u \in H$. Fix $\varepsilon > 0$. Pick an $N$ such that $\sum_{n>N} |\langle \varphi_n, u \rangle|^2 < \varepsilon$. Then, using Parseval we find that

$$\limsup_{p \to \infty} ||A(1/p)u - u||^2$$

$$= \limsup_{p \to \infty} \left( \sum_{n=1}^{N} |\lambda_n(1/p) - 1|^2 |\langle \varphi_n, u \rangle|^2 + \sum_{n=N+1}^{\infty} |\lambda_n(1/p) - 1|^2 |\langle \varphi_n, u \rangle|^2 \right)$$

$$\leq \sum_{n=1}^{N} (\limsup_{p \to \infty} |\lambda_n(1/p) - 1|^2) |\langle \varphi_n, u \rangle|^2 + 2 \sum_{n=N+1}^{\infty} |\langle \varphi_n, u \rangle|^2 < 2\varepsilon.$$

Since $\varepsilon$ was arbitrary, it follows that $\lim_{p \to \infty} ||A_p u - u|| = 0$.

To prove that $A_p$ cannot converge in norm to $I$, simply pick for any $p > 0$, an $n \in \mathbb{N}$ such that $|\lambda_n(1/p) - 1| \geq 1/2$. Then

$$||A_p - I|| = \sup_{||u|| = 1} ||A_p u - u|| \geq ||A_p \varphi_n - \varphi_n|| = ||(\lambda_n(1/p) - 1) \varphi_n|| \geq 1/2.$$
Problem 5: Consider the Hilbert space $H = L^2(\mathbb{T})$, and the operator $A \in B(H)$ defined by $[A u](x) = (1 + \cos x) u(x)$. Prove that $A$ is self-adjoint and positive, but not coercive. (5p)

Set $\varphi(x) = 1 + \cos(x)$.

That $A$ is self-adjoint follows immediately from the fact that $1 + \cos x$ is real:

$$
\langle A u, v \rangle = \int_{-\pi}^{\pi} (1 + \cos x) u(x) v(x) \, dx = \int_{-\pi}^{\pi} \overline{u(x)} \left((1 + \cos x) v(x)\right) \, dx = \langle u, A v \rangle.
$$

That $A$ is non-negative follows from the fact that $1 + \cos x$ is non-negative:

$$
(2) \quad \langle A u, u \rangle = \int_{-\pi}^{\pi} (1 + \cos x)|u(x)|^2 \, dx \geq 0.
$$

To further prove that $A$ is positive, note that if we have equality in (2), then $u(x)$ must be zero everywhere except possibly on a set of measure zero, since $1 + \cos x$ is zero only for $x = \pm \pi$.

Recall that $A$ is coercive iff

$$
\inf_{||u||=1} \langle A u, u \rangle > 0.
$$

To prove that this is not true, define the functions $u_n \in H$ by

$$
u_n(x) = \begin{cases} 
\sqrt{n} & x \in [\pi - 1/n, \pi], \\
0 & x \in (-\pi, \pi - 1/n).
\end{cases}
$$

Note that $||u_n|| = 1$, so

$$
\inf_{||u||=1} \langle A u, u \rangle \leq \inf_{n \in \mathbb{N}} \langle A u_n, u_n \rangle = \inf_{n \in \mathbb{N}} \int_{\pi - 1/n}^{\pi} (1 + \cos x) |u_n(x)|^2 \, dx
\leq \inf_{n \in \mathbb{N}} \int_{\pi - 1/n}^{\pi} (1 + \cos(\pi - 1/n)) n \, dx = \inf_{n \in \mathbb{N}} \left(1 + \cos(\pi - 1/n)\right) = 0.
$$
**Problem 6:** Consider the Hilbert space \( H = L^2(\mathbb{R}) \). For this problem, we define \( H \) as the closure of the set of all compactly supported smooth functions on \( \mathbb{R} \) under the norm
\[
||u|| = \left( \int_{-\infty}^{\infty} |u(x)|^2 \, dx \right)^{1/2}.
\]
Which of the following sequences converge weakly in \( H \)? Motive your answers briefly.
(2p each)

(a) \((u_n)_{n=1}^\infty\) where \( u_n(x) = \begin{cases} |x - n|, & \text{for } x \in [n - 1, n + 1], \\ 0, & \text{for } x \in (-\infty, n - 1) \cup (n + 1, \infty). \end{cases} \)

(b) \((v_n)_{n=1}^\infty\) where \( v_n(x) = \sin(nx) e^{-x^2} \).

(c) \((w_n)_{n=1}^\infty\) where \( w_n(x) = e^{-x^2/n} \).

**Remark:** Note that there exist functions \( f \) and \( f_n \) in \( H \) such that 
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) f_n(x) \, dx \neq \int_{-\infty}^{\infty} f(x) \left( \lim_{n \to \infty} f_n(x) \right) \, dx.
\]
Keeping in mind the definition of \( H \) given above, you can solve the problem without having to make such interchanges (not using any Lebesgue integrals at all).

Recall that if a sequence \((\varphi_n)_{n=1}^\infty\) is bounded, and there exists a function \( \varphi \in H \) such that 
\[
\langle \varphi_n, \psi \rangle \to \langle \varphi, \psi \rangle \text{ for all } \psi \text{ in a dense subset } \mathcal{P},
\]
then \( \varphi_n \rightharpoonup \varphi \). In (a) and (b), we let \( \mathcal{P} \) be the set of compactly supported smooth functions (this is dense by definition).

(a) Since \( u_n(x) = u_1 (x - n + 1) \), it follows that \( ||u_n|| = ||u_1|| \) and so \((u_n)\) is a bounded sequence. Furthermore, if \( \psi \in \mathcal{P} \), then \( \langle u_n, \psi \rangle \to 0 \) since for large enough \( n \), the support of \( u_n \) will be outside the support of \( \psi \). It follows that \( u_n \rightharpoonup 0 \).

(b) \( ||v_n||^2 = \int_{-\infty}^{\infty} |\sin(nx)|^2 e^{-2x^2} \, dx \leq \int_{-\infty}^{\infty} e^{-2x^2} \, dx \) so \((v_n)\) is bounded. Furthermore, if \( \psi \in \mathcal{P} \), then
\[
|\langle v_n, \psi \rangle| = \left| \int_{-\infty}^{\infty} \sin(nx) e^{-x^2} \psi(x) \, dx \right| = \{\text{partial integration}\}
\]
\[
= \left| \int_{-\infty}^{\infty} \frac{1}{n} \cos(nx) \frac{d}{dx} \left(e^{-x^2} \psi(x)\right) \, dx \right| \leq \frac{1}{n} \int_{-\infty}^{\infty} \left| \frac{d}{dx} \left(e^{-x^2} \psi(x)\right) \right| \, dx \to 0,
\]
so \( v_n \rightharpoonup 0 \) (the boundary terms vanish since \( \psi \) has compact support).

(c) \( ||w_n||^2 = \int_{-\infty}^{\infty} e^{-2x^2/n} \, dx = \{x = \sqrt{n}y\} = \sqrt{n} \int_{-\infty}^{\infty} e^{-2y^2} \, dy = \sqrt{n} ||w_1||^2 \to \infty \)
so \((w_n)\) cannot converge weakly.