Homework set 8 — APPM5450, Spring 2007 — Hints

11.5: Note that
\[
\frac{x + i\varepsilon}{\varepsilon^2 + x^2} = \frac{x}{\varepsilon^2 + x^2} - \frac{i\varepsilon}{\varepsilon^2 + x^2}.
\]
Fix a \(\varphi \in S\). You need to prove that

(1) \[\lim_{\varepsilon \to 0} \langle i\frac{\varepsilon}{\varepsilon^2 + x^2}, \varphi \rangle \to -i\pi\varphi(0).\]

and that

(2) \[\lim_{\varepsilon \to 0} \langle \frac{x}{\varepsilon^2 + x^2}, \varphi \rangle \to \langle \text{PV}\left(\frac{1}{x}\right), \varphi \rangle,\]

Proving (1) is simple:
\[
\langle i\frac{\varepsilon}{\varepsilon^2 + x^2}, \varphi \rangle = \int_{-\infty}^{\infty} i\frac{\varepsilon}{\varepsilon^2 + x^2} \varphi(x) \, dx = \{\text{Set } x = \varepsilon y = \ldots\}
\]

For (2) we need to work a bit more (unless I overlook a simpler solution)
\[
\lim_{\varepsilon \to 0} \langle \frac{x}{\varepsilon^2 + x^2}, \varphi \rangle - \langle \text{PV}\left(\frac{1}{x}\right), \varphi \rangle
\]
\[
= \lim_{\varepsilon \to 0} \int_{\varepsilon \geq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} \varphi(x) \, dx - \lim_{\varepsilon \to 0} \int_{|x| \geq \sqrt{\varepsilon}} \frac{1}{x} \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_{|x| \geq \sqrt{\varepsilon}} \left(\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x}\right) \varphi(x) \, dx + \lim_{\varepsilon \to 0} \int_{|x| \leq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} \varphi(x) \, dx.
\]

First we bound \(|S_1|\). Note that when \(|x| \geq \sqrt{\varepsilon}\), we have
\[
\left|\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x}\right| \leq \varepsilon^2 |x|^{-3} \leq \varepsilon^2 \varepsilon^{-3/2} = \varepsilon.
\]
Consequently,
\[
|S_1| \leq \limsup_{\varepsilon \to 0} \int_{|x| \geq \sqrt{\varepsilon}} \left|\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x}\right| |\varphi(x)| \, dx
\]
\[
\leq \limsup_{\varepsilon \to 0} \int_{|x| \geq \sqrt{\varepsilon}} \varepsilon \left(1 + |x|^2\right) |(1 + |x|^2)| \varphi(x) |x| \leq ||\varphi||_{0,2} dx = 0.
\]

In bounding \(S_2\) we use that
\[
\int_{|x| \leq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} \varphi(0) \, dx = 0,
\]
and that
\[
|\varphi(x) - \varphi(0)| \leq |x||\varphi'||_{u} \leq |x||\varphi||_{1,0},
\]
to obtain
\[ |S_2| = \left| \lim_{\varepsilon \to 0} \int_{|x| \leq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} \left( \varphi(x) - \varphi(0) \right) dx \right| \leq \limsup_{\varepsilon \to 0} \int_{|x| \leq \sqrt{\varepsilon}} \frac{|x|}{\varepsilon^2 + x^2} \left| x \right| ||\varphi||_{1,0} dx = 0. \]

**Problem 11.6:** We find that
\[ \langle D(\log |x| \varphi) \rangle = -\langle \log |x| \varphi' \rangle = -\int_\mathbb{R} \log |x| \varphi'(x) dx = -\lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{-\varepsilon} \log(-x)\varphi'(x) dx + \int_{\varepsilon}^{\infty} \log(x)\varphi'(x) dx \right]. \]

Now simply perform partial integration in each term separately.

**Problem 11.7:** First prove that \( x \cdot \delta(x) = 0 \) and that \( x \cdot \text{PV}(1/x) = 1 \) (using the regular rules for the product between a polynomial and a Schwartz function). Suppose that \( \cdot \) is distributive and can pair any two distributions. Then on the one hand we would have
\[ \delta(x) \cdot x \cdot \text{PV}(1/x) = \delta(x) \cdot (x \cdot \text{PV}(1/x)) = \delta(x) \cdot 1 = \delta(x). \]

But we would also have
\[ \delta(x) \cdot x \cdot \text{PV}(1/x) = (x \cdot \delta(x)) \cdot \text{PV}(1/x) = 0 \cdot \text{PV}(1/x) = 0. \]

This is a contradiction.

**Problem 11.8:** Fix \( \varphi \in \mathcal{S} \). Set \( \alpha = \int \varphi \), and define
\[ (3) \quad \psi(x) = \int_{-\infty}^{x} (\varphi(z) - \alpha \omega(z)) dz. \]

Obviously, \( \psi \in C^\infty \), and
\[ (4) \quad \varphi(x) = \alpha \omega(x) + \psi'(x). \]

Moreover, we find that if \( n \geq 1 \), then
\[ ||\psi||_{n,k} = ||(1 + |x|^2)^{k/2} \psi^{(n)}||_u \]
\[ = ||(1 + |x|^2)^{k/2} (\varphi^{(n-1)} - \alpha \omega^{(n-1)})||_u \leq ||\varphi||_{n-1,k} + |\alpha| ||\omega||_{n-1,k}. \]

It remains to prove that for any \( k \),
\[ \sup_x (1 + |x|^2)^{k/2} |\psi(x)| < \infty. \]
First consider \( x \leq 0 \). Then for any \( k \), we have
\[
\sup_{x \leq 0} (1 + |x|^2)^{k/2} |\psi(x)| 
\leq \limsup_{x \leq 0} \left[ (1 + |x|^2)^{k/2} \int_{-\infty}^{x} \frac{1}{(1 + |y|^{(k+2)/2})^2} ||\varphi||_{0,k+2} \, dy \right] 
+ |\alpha| (1 + |x|^2)^{k/2} \int_{-\infty}^{x} \frac{1}{(1 + |y|^{(k+2)/2})^2} ||\omega||_{0,k+2} \, dy < \infty.
\]
To prove the corresponding estimate for \( x \geq 0 \), we use that since
\[
\int_{-\infty}^{x} (\varphi(z) - \alpha \omega(z)) \, dz + \int_{x}^{\infty} (\varphi(z) - \alpha \omega(z)) \, dz = 0,
\]
we can also express \( \psi \) as
\[
\psi(x) = -\int_{x}^{\infty} (\varphi(z) - \alpha \omega(z)) \, dz.
\]
Then proceed as in the bound for \( x \leq 0 \).

**Problem 1:**
\[
\langle D f, \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-\infty}^{0} (-x) \varphi'(x) \, dx - \int_{0}^{\infty} x \varphi'(x) \, dx
\]
\[
= \left[ x \varphi(x) \right]_{0}^{\infty}_{-\infty} - \int_{-\infty}^{0} \varphi(x) \, dx - \left[ x \varphi(x) \right]_{0}^{\infty} + \int_{-\infty}^{0} \varphi(x) \, dx = \langle g, \varphi \rangle,
\]
where
\[
g(x) = \begin{cases} 
-1 & x \leq 0 \\
1 & x > 0.
\end{cases}
\]
So \( D f = g \). (Note that the value of \( g(0) \) is irrelevant, any finite value can be assigned.) To compute \( D^2 f \), simply differentiate \( g \) in the same way. You should find that \( D^2 f = 2\delta \).

**Problem 2:** This is a fairly straight-forward application of the definitions.

**Problem 3:** Define for \( n = 1, 2, 3, \ldots \), the functions
\[
\chi_n(x) = \begin{cases} 
1 & x \in \left[n - \frac{1}{4\pi}, n\right], \\
0 & \text{otherwise},
\end{cases}
\]
and set
\[
f(x) = \sum_{n=1}^{\infty} 2^n \chi_n(x).
\]
Now prove that both (2) and (3) hold for any \( k \).