Homework set 6 — APPM5450, Spring 2007

If you didn't complete all of the problems 9.1 – 9.11 last week, then continue working on that.

Problem 1: Let $H_1$ and $H_2$ be Hilbert spaces, let $U : H_1 \to H_2$ be unitary, and let $A \in \mathcal{B}(H_1)$. Define $\tilde{A} \in \mathcal{B}(H_2)$ by $\tilde{A} = U^{-1}AU$. Prove that

- $\sigma_p(A) = \sigma_p(\tilde{A})$
- $\sigma_c(A) = \sigma_c(\tilde{A})$
- $\sigma_r(A) = \sigma_r(\tilde{A})$

Problem 2: Let $A$ be a self-adjoint compact operator. For $\lambda \in \rho(A)$, set $R_\lambda = (A - \lambda I)^{-1}$ as usual. Construct the spectral decomposition of $R_\lambda$. Use it to prove that

\[
||R_\lambda|| = \frac{1}{\text{dist}(\lambda, \sigma(A))} = \frac{1}{\inf_{\mu \in \sigma(A)} |\lambda - \mu|}.
\]

Problem 3: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi, \pi]$. Define

\[
\Omega_t = \{ u \in H : u(x) = 0 \forall x \geq t \}.
\]

Note that $\Omega_t$ is a closed linear subspace of $H$. Define $P(t)$ as the orthogonal projection onto $\Omega_t$. Consider the operator $A \in \mathcal{B}(H)$ defined by

\[
[Au](x) = x u(x).
\]

(a) Prove that $\Omega_t$ is an invariant subspace of $A$ for every $t \in \mathbb{R}$.

(b) Prove that if $a < b \leq c < d$, then \(\text{ran}(P(b) - P(a)) \perp \text{ran}(P(d) - P(c))\). Conclude that for any numbers $-\pi = t_0 < t_1 < t_2 < \cdots < t_n = \pi$, it is the case that

\[H = \text{ran}[P(t_1) - P(t_0)] \oplus \text{ran}[P(t_2) - P(t_1)] \oplus \cdots \oplus \text{ran}[P(t_n) - P(t_{n-1})],\]

where each term is an invariant subspace of $A$.

(c) For a positive integer $n$, set $h = 2\pi/n$, and $\lambda_j = -\pi + h j$. Define the operator

\[
A_n = \sum_{j=1}^{n} \lambda_j \left(P(\lambda_j) - P(\lambda_{j-1})\right).
\]

Prove that $||A - A_n|| \leq 2\pi/n$. Conclude that $A_n \to A$ in norm.

Note: The sum (1) is a Riemann-Stieltjes sum of the integral $A = \int_I \lambda dP(\lambda)$. 

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