Homework set 3 — APPM5450, Spring 2007

From the textbook: 8.4, 8.6. Optional: 8.5.

Problem 1: Let $H$ be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for $H$. Given a bounded sequence of complex number $(\lambda_n)_{n=1}^{\infty}$, define the operator $A$ by setting $Au = \sum_{n=1}^{\infty} \lambda_n \varphi_n \langle \varphi_n, u \rangle$.

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Prove that $A^* u = \sum_{n=1}^{\infty} \bar{\lambda}_n \varphi_n \langle \varphi_n, u \rangle$. Conclude that $A$ is self-adjoint iff all $\lambda_n$’s are real. When is $A$ skew-symmetric?

Problem 2: Consider the Hilbert space $H = L^2([-\pi, \pi])$, and the operator $A \in B(H)$ defined by $[Au](x) = |x| u(x)$. Prove that $A$ is self-adjoint and positive, but not coercive. Prove that $\langle u, v \rangle_A = \langle A u, v \rangle$ is an inner product on $H$, but that the topology generated by (the norm generated by) this inner product is not equivalent to the topology generated by the $L^2$-norm.

Problem 3: Set $H = l^2(\mathbb{Z})$ and let $R$ denote the right-shift operator (so that if $y = Rx$, then $y_n = x_{n-1}$). Construct $R^*$. Prove that $R$ is unitary. (Recall that the right-shift operator on $l^2(\mathbb{N})$ is not unitary!)

Problem 4: Consider the Hilbert space $L^2(\mathbb{T})$. Let $k$ denote a continuous function on $\mathbb{T}$ that takes on complex values. Let $A$ denote the operator $[Au](x) = \int_{\mathbb{T}} k(x, y) u(y) \, dy$. Prove that $[A^* u](x) = \int_{\mathbb{T}} \bar{k}(y, x) u(y) \, dy$. Conclude that $A$ is self-adjoint iff $k(x, y) = \bar{k}(y, x) \ \forall \ x, y \in \mathbb{T}$.

Problem 5 (optional): Assume that you have proved that in a Hilbert space, ran$(I - K)$ is closed whenever $K$ has finite rank. Prove from this that ran$(I - K)$ is closed for all compact operators $K$. (Then if you feel industrious, feel free to try proving the statement given in the first sentence.)