Problem 1: No motivation required for these questions. 2p each.

(a) State Hölder’s inequality.

(b) Define what it means for a sequence \((\varphi_n)_{n=1}^\infty\) of Schwartz functions to converge in \(\mathcal{S}(\mathbb{R})\).

(c) Let \(H\) be a Hilbert space, and let \((A_n)_{n=1}^\infty\) be a sequence of operators in \(\mathcal{B}(H)\). Define what it means for \(A_n\) to converge strongly to some operator \(A \in \mathcal{B}(H)\).

(d) Let \((X, \mu)\) be a \(\sigma\)-finite measure space. For which numbers \(p\) in the interval \([1, \infty]\) is it necessarily the case that \((L^p(X, \mu))^* = L^q(X, \mu)\), where \(q\) is such that \((1/p) + (1/q) = 1\). For which numbers \(p\) is \(L^p(X, \mu)\) necessarily reflexive?

(e) Let \(H\) be a Hilbert space, and let \(A\) be a linear bounded operator on \(H\). Give a formula that relates the range of \(A\) to the kernel of \(A^*\).

(f) Let \(H\) be a Hilbert space and let \(A \in \mathcal{B}(H)\) be a self-adjoint operator. Let \(H_1\) be an invariant subspace of \(A\). Is \(H_1^\perp\) necessarily an invariant subspace of \(A\)? Is \(H_1^\perp\) necessarily an invariant subspace of \(A\) if \(A\) is skew-adjoint instead of self-adjoint?

(g) Let \(H\) be a Hilbert space, and let \(A \in \mathcal{B}(H)\) be self-adjoint and compact. What can you say about \(\sigma_c(A)\)?

Problem 2: Let \(H\) be a Hilbert space, and let \(P \in \mathcal{B}(H)\) be an operator such that \(P^2 = P\). Prove that the statements (S1) and (S2) given below are equivalent: (4p)

(S1): \((\text{ran}(P))^\perp = \ker(P)\).

(S2): \(\langle Px, y \rangle = \langle x, Py \rangle\) for all \(x, y \in H\).

Problem 3: Let \(\delta \in \mathcal{S}(\mathbb{R})^*\) denote the Dirac delta-function as usual, let \(\delta'\) denote the distributional derivative of \(\delta\), and define for a positive integer \(n\) the distribution \(T_n \in \mathcal{S}(\mathbb{R})^*\) by \(T_n(x) = \sin(nx) \delta'(x)\).

(a) Calculate the Fourier transform \(\hat{T}_n\) of \(T_n\). (2p)

(b) Does the sequence \((\hat{T}_n)_{n=1}^\infty\) converge in \(\mathcal{S}(\mathbb{R})^*\)? (2p)

Hint: You may want to start by simplifying the expression for \(T_n\).
**Problem 4:** Let \( p \in [1, \infty) \), let \( g \) be a function in \( L^p(\mathbb{R}) \), and let \((f_n)_{n=1}^{\infty}\) be measurable functions from \( \mathbb{R} \to \mathbb{R} \) such that
\[
\sum_{n=1}^{\infty} |f_n(x)| \leq g(x), \quad \text{a.e.}
\]
Set \( h_N = \sum_{n=1}^{N} f_n \). Prove that the sequence \((h_N)_{N=1}^{\infty}\) converges in \( L^p(\mathbb{R}) \). (5p)

**Problem 5:** Consider the Hilbert space \( H = L^2(\mathbb{T}) \), let \( a \in (0, \pi) \) be a real number, and define the operator \( T \in B(H) \) by
\[
[Tu](x) = \frac{1}{2} (u(x-a) + u(a-x)).
\]

(a) Construct \( T^* \) and indicate whether \( T \) is self-adjoint. (2p)

(b) Prove that \( T \) is not unitary. Is \( T \) normal? (2p)

(c) Specify infinite dimensional subspaces \( H_1 \) and \( H_2 \) of \( H \) such that the map \( T : H_1 \to H_2 \) is a unitary operator. (2p)

(d) Let \( \mathcal{F} : H \to l^2(\mathbb{Z}) \) denote the Fourier transform. Determine the operator \( \hat{T} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) given by \( \hat{T} = \mathcal{F} T \mathcal{F}^{-1} \). (2p)

(e) Determine \( \sigma(T) \). As far as you can, classify the different parts of the spectrum as belonging to the point, continuous, or residual spectrum. (3p)

*Hint:* You may want to attempt question 5(e) last as it could be time-consuming.