Applied Analysis (APPM 5450): Midterm 3
5.00pm – 6.20pm, Apr 24, 2006. Closed books.

Note: In your solutions, explicitly state if you use an integral sign that does not refer to a Lebesgue integral. (All integrals on this page are Lebesgue integrals.)

Problem 1: In this question, \((X, \mathcal{A}, \mu)\) denotes a measure space.

(a) What axioms must \(\mathcal{A}\) satisfy? (2p)

(b) What axioms must \(\mu\) satisfy? (2p)

(c) Prove that if \(\Omega_1, \Omega_2 \in \mathcal{A}\), then \(\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)\). Given an exact condition for when equality occurs. (“Equality occurs if and only if . . . ”) (2p)

(d) Define the Lebesgue integral of a measurable non-negative function on \(X\). (2p)

(e) Define the “essential supremum” of a measurable function \(f\) on \(X\). (2p)

(f) For which extended real numbers \(p\) are the simple functions dense in \(L^p(\mathbb{R}^d)\)? When is \(C_0^\infty(\mathbb{R}^d)\) dense in \(L^p(\mathbb{R}^d)\)? (2p)

Problem 2: Define the real-valued function \(f\) on \(\mathbb{R}^2\) by \(f(x_1, x_2) = x_1 x_2^2\). Define a tempered distribution \(T_f\) by \(\langle T_f, \varphi \rangle = \int_{\mathbb{R}^2} f(x) \varphi(x) \, dx\). What is the Fourier transform of \(T_f\)? Motivate your answer carefully. (4p)

Problem 3: Calculate the limit
\[
\lim_{n \to \infty} \int_n^{n+1} \sqrt{x} \tan(1/\sqrt{x}) \, dx.
\]
Motivate your answer carefully. (4p)

Problem 4: Recall that for \(s \in [0, \infty)\), the Sobolev space \(H^s(\mathbb{R}^d)\) is defined as the set of all functions \(f \in L^2(\mathbb{R}^d)\) such that \((1 + |t|^2)^{s/2} \hat{f}(t) \in L^2\). Prove that if \(s\) is large enough, then \(H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)\). (4p)
Problem: The following problems are worth 2p each.

(a) State the definition of a $\sigma$-algebra.

(b) State the definition of a measure.

(c) Let $(X, A, \mu)$ denote a measure space. Suppose that $\Omega_1, \Omega_2 \in A$. Prove that $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$. Give a condition for when equality occurs.

Problem: Given a measure space $(X, A, \mu)$, define (a) the Lebesgue integral of a simple function, and (b), the Lebesgue integral of a measurable non-negative function. (4p)

Problem: Assume that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in $\mathcal{S}(\mathbb{R})$. Set $\chi_n(x) = \varphi_n(x) \psi_n(x)$. Prove that $\chi_n \to \varphi \psi$ in $\mathcal{S}$. 