Note: The problems are worth two points each, for a total of 16 points.

Problem 1: In this problem, $\partial = (d/dx)$, and $\delta \in S^*(\mathbb{R})$ denotes the Dirac delta function.

(a) For $T \in S^*(\mathbb{R})$, define $\partial T$, and prove that what you define is a continuous functional on $S(\mathbb{R})$. (You may use the fact that $\partial : S \to S$ is continuous.)

(b) Set $U(x) = x [\partial \delta](x)$, and calculate, for $\varphi \in S$, $\langle U, \varphi \rangle$.

(c) Set $V(x) = x \delta(x)$, and calculate, for $\varphi \in S$, $\langle \partial V, \varphi \rangle$.

Problem 2: We define the functions $\varphi_n \in S$ by setting

$$\varphi_n(x) = \sqrt{x^2+1/n} e^{-x^2/\sqrt{x^2+1/n}}.$$ 

Does the sequence converge in $S$ as $n \to \infty$? If so, to what?

Problem 3: Let $H$ be a Hilbert space and let $A$ be a compact self-adjoint operator on $H$. Let $b$ be a non-zero real number, and set $f(x) = (x - i b)^{-1}$ where $i$ is the imaginary unit. This question concerns different ways of defining $f(A)$.

(a) Noting that $f$ has the MacLaurin expansion $f(x) = (-1/ib) \sum_{n=0}^{\infty} (x/ib)^n$, we define $B_N = (-1/ib) \sum_{n=1}^{N} ((1/ib) A)^n$. Describe when, if ever, the sequence $(B_N)_{N=1}^{\infty}$ converges in norm in $B(H)$.

(b) Let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for $H$ consisting of eigenvectors of $A$, so that $A \varphi_n = \lambda_n \varphi_n$. Define the operator $C_N$ by setting, for $u \in H$, $C_N u = \sum_{n=1}^{N} f(\lambda_n) (\varphi_n, u) \varphi_n$. Describe when, if ever, the sequence $(C_N)_{N=1}^{\infty}$ converges strongly in $B(H)$.

(c) Describe when, if ever, the sequence $(C_N)_{N=1}^{\infty}$ converges in norm in $B(H)$.

Problem 4: Let $R$ denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n \cos(nx) & \text{for } |x| \leq R, \\ 0 & \text{for } |x| > R. \end{cases}$$

For which numbers $R$, if any, is it the case that $f_n \to 0$ in $S^*$?