Problem 1: Let $H$ be a Hilbert space, and let $(e_j)_{j=1}^n$ be an orthonormal set in $H$. Let $x$ be an arbitrary vector in $H$. Set $M = \text{span}(e_1, \ldots, e_n)$, set
\[ y = \sum_{j=1}^n (e_j, x) e_j, \]
and set $z = x - y$. Prove that $z \in M^\perp$ (and consequently, that $y \perp z$). Prove that
\[ ||x - y|| = \inf_{y' \in M} ||x - y'||. \]
Prove that $y$ is the unique minimizer (in other words, if $y' \in M \setminus \{y\}$, then $||x - y'|| > ||x - y||$). Prove these claims directly, without using the theorem about existence of a unique minimizer between a closed convex set and a point.

Problem 2: Set $I = [-1, 1]$ and consider the Hilbert space $H = L^2(I)$. Let $M$ denote the subspace of $H$ consisting of all even functions (in other words, functions such that $f(x) = f(-x)$ for all $x$). Given an $f \in H$, prove that
\[ \inf_{g \in M} ||f - g|| = \left( \int_{-1}^1 \left| \frac{f(x) - f(-x)}{2} \right|^2 \, dx \right)^{1/2}. \]
(Don’t worry about issues relating to Lebesgue integration.)

Problem 3: Let $X$ be a separable infinite-dimensional Hilbert space. Prove that there exists a family of closed linear subspaces $\{\Omega_t : t \in [0, 1]\}$ such that $\Omega_s$ is a strict subset of $\Omega_t$ whenever $s < t$.

Hint: It might be easier to solve the problem if you consider a particular Hilbert space, such as, e.g., $H = L^2(I)$, for $I = [0, 1]$. If you can solve the problem for this specific $H$, you can then invoke the theorem that all separable Hilbert spaces are unitarily equivalent.