Problem 4.1: Let $K$ be a compact set and fix $x \notin K$. To prove that $K^c$ is open (which is to say that $K$ is closed) we need to construct an open set $G$ such that $x \in G \subseteq K^c$.

For every $y \in K$, pick disjoint open sets $G_y \ni x$ and $H_y \ni y$. Then $\{H_y\}_{y \in K}$ forms an open cover of $K$. Since $K$ is compact, there is a finite subcover $\{H_{y_j}\}_{j=1}^J$. Now set $G = \bigcap_{j=1}^J G_{y_j}$.

Clearly $G$ is open (since its a finite intersection of open sets) and $x \in G$. Since $G$ is also disjoint from every set $H_{y_j}$ in the finite open cover of $K$, it follows that $G \cap K = \emptyset$ (which is to say $G \subseteq K^c$).

Problem 4.2: For the cantor set $C$ we have $\overline{C} = \partial C = C$ and $C^o = \emptyset$.

Problem 4.3: Just apply the definitions.

Problem 4.5a: The connected subspaces of $\mathbb{R}$ are the intervals.

Proof that any interval is connected: Let $I$ be an interval, and suppose that $I = G_1 \cup G_2$ where $G_1$ and $G_2$ are both open, and $G_1 \cap G_2 = \emptyset$.

Pick a point $t \in I$. The point belongs to either $G_1$ or $G_2$. Say $t \in G_1$. Our claim is that then $G_2 \cap (t, \infty)$ must be empty. Suppose not, then set $s = \inf G_2 \cap (t, \infty)$. Since $G_1$ is open, it is not possible that $s \in G_1$ (if $s \in G_1$, there would be $\varepsilon > 0$ such that $B_{\varepsilon}(t) \subset G_1$ and then $\inf G_2 \cap (t, \infty) \geq s + \varepsilon$). Since $G_2$ is open, it not possible that $s \in G_2$ (if $s \in G_2$, there would be $\varepsilon > 0$ such that $B_{\varepsilon}(t) \subset G_2$ and then $\inf G_2 \cap (t, \infty) \leq s - \varepsilon$). Therefore $s \notin I$, which is impossible.

The proof that $G_2 \cap (-\infty, t)$ must be empty is analogous. It follows that $G_2$ must be empty.

Proof that any non-interval is not connected: Let $I$ be a subset of $\mathbb{R}$ that is not an interval. Then there is a point $t \notin I$ such that the two sets

$G_1 = (-\infty, t) \cap I, \quad G_2 = (t, \infty) \cap I$

are both non-empty. Both $G_1$ and $G_2$ are open in the subspace topology (by definition), they are non-intersecting, and $I = G_1 \cup G_2$.

Problem 1: Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and define $f : X \to Y$ by setting $f([x_1, x_2]) = x_1$. Prove that $f$ is continuous. Prove that $f$ is open. Prove that $f$ does not necessarily map closed sets to closed sets.

Hint: That $f$ is continuous is very easy to prove.

Let $G$ be an open set in $X$. Pick $t \in f(G)$. There is some $x \in G$ such that $t = f(x)$. Since $G$ is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq G$. But then $(t - \varepsilon, t + \varepsilon) = f(B_\varepsilon) \subseteq f(G)$ so $f(G)$ is open.
Consider the set \( F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 = 1\} \). Then \( F \) is closed. But \( f(F) = (-\infty, 0) \cup (0, \infty) \) which is not closed.

**Problem 2:** Prove that the co-finite topology is first countable if and only if \( X \) is countable.

*Hint:* Read the definitions carefully — there is nothing tricky about this question.

**Problem 3:** Prove that the co-finite topology on \( \mathbb{R} \) weaker than the standard topology.

*Hint:* Note that all you need to do is to demonstrate that any set that is open in the cofinite topology is also open in the standard topology.